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# Matter as Spectrum of Spacetime Representations

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Bound and scattering state Schrödinger functions of nonrelativistic quantum mechanics as representation matrix elements of space and time are embedded into residual representations of spacetime as generalizations of Feynman propagators. The representation invariants arise as singularities of rational representation functions in the complex energy and complex momentum plane. The homogeneous space $\mathbf{GL}(\mathbb{C})^2/\mathbf{U}(2)$ with rank 2, the orientation manifold of the unitary hypercharge-isospin group, is taken as model of nonlinear spacetime. Its representations are characterized by two continuous invariants whose ratio will be related to gauge field coupling constants as residues of the related representation functions. Invariants of product representations define unitary Poincaré group representations with masses for free particles in tangent Minkowski spacetime.

**KEY WORDS:** matter; spacetime representations.

## 1. INTRODUCTION

In Wigner's classification (Wigner, 1939) of the unitary irreducible Poincaré group representations the particles are characterized by two invariants—a mass $m^2$ for translations and a spin (polariziation) $J$ for rotations. Therewith, linear spacetime and free particles originate from one operational concept, from a group and its representations. Why the free particles have the observed masses, spins, and charges $z$ for the additional internal operations, that is not explained by classifying the representations of linear spacetime. The actual spectrum of matter $(m^2, 2J, z) \in \mathbb{R} \times \mathbb{N} \times \mathbb{Z}$ has to be understood by additional structures, e.g., by representation invariants of nonlinear spacetime. A related attempt is given in this paper.

The representation classes of the additive group $\mathbb{R}^d$ (translations) are its characters—energies for time translations $\mathbb{R}$ and momenta for position translations $\mathbb{R}^3$. The translation characters constitute the dual group $\check{\mathbb{R}}^d$ (dual space) and give rise to convolution algebras of energy and momentum distributions and

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functions. A homogeneous spacetime manifold with tangent Minkowski translations $x \in \mathbb{R}^4$ is representable by residues (Saller, 2001a) of Fourier transformed energy–momentum $q \in \check{\mathbb{R}}^4$ distributions. The representation characterizing invariants arise as poles in the complex energy and complex momentum plane. Product representations come with convoluted energy–momentum distributions and functions.

In Feynman propagators (Saller, 1997a) as tempered distributions, the Dirac energymomentum distributions on the mass shell $\vartheta(\pm q_0)\delta(q^2 - m^2)$ describe free particles, acted upon by unitary representations of the Poincaré group, e.g., $e^{iq_0 t}\frac{\sin|\vec{q}|r}{r}$, $q_0^2 - \vec{q}^2 = m^2$. The principal value distribution $\frac{1}{q_{\rm P}^2 - m^2}$ describes also interactions, e.g., Yukawa interactions in $e^{iq_0 t}\frac{e^{-|Q|r}}{r}$, $q_0^2 + Q^2 = m^2$. In Feynman integrals as convolutions of energy–momentum distributions the on-shell parts with the matrix elements of unitary spacetime translation representations give product representation matrix elements, i.e., products of free states. The causally supported parts with the off-shell contributions, i.e., the Yukawa interactions with nonunitary position representations, are not convolutable. This is the origin of the "divergence" problem in quantum field theories with interactions.

Representations of spacetime embed time and position representations. The compact time representations induce (Folland, 1995; Mackey, 1968; Wigner, 1939) compact representations of spacetime translations, related to free particles. The noncompact position representations[2] as seen in Hilbert space valued Schrödinger functions, e.g., $e^{-|m|r} = \int \frac{d^3q}{\pi^2}\frac{|m|}{(\vec{q}^2+m^2)^2}e^{-i\vec{q}\vec{x}}$, induce Lorentz compatible representations of the spacetime translation future cone that is taken as model of nonlinear spacetime (Saller, 1997b, 1999, 2001b). The position representations are embedded into causally supported contributions. Those parts do not describe free particles, they are used for wave functions of particles as their "inner structure." The invariant mass for the representation of the position degree of freedom comes in a higher order pole, e.g., $\frac{1}{(q^2-m^2)^2}$. The representation invariant cannot be interpreted as a mass for a free particle.

After the discussion of time representations (harmonic oscillator), position representations (Schrödinger wave functions), and spacetime translation representations (Feynman propagators), all in the language of residual representations with rational complex functions, representations of nonlinear spacetime are given and an attempt is made to derive particles as product representations of spacetime.

In the following, I have included, for better readability, many familiar explicit calculation. The special functions are used as given in the book of Vilenkin and Klimyk (1991).

[2] Some people find it surprising that unitary representations $e^{imt} \in \mathcal{S}'(\mathbb{R})$ are no elements of a Hilbert space with its unitary product in contrast to nounitary representations $e^{-|mz|} \in L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$.

## 2. RESIDUAL REPRESENTATIONS OF SYMMETRIC SPACES

Representation matrix elements of a real finite dimensional symmetric space $G/H$ with a Lie subgroup $H \subseteq G$ are complex functions thereon

$$g : (G/H)_{\text{repr}} \to \mathbb{C}, \quad x \mapsto g(x)$$
$$k \in G : g_k(x) = g(k \bullet x)$$

The symmetric space is assumed to have a *canonical parametrization* by an orbit in a real vector space V

$$x \in G \bullet x_0 \cong G/H, \quad G \bullet x_0 \subseteq V \cong \mathbb{R}^d$$

e.g., a group by its Lie algebra $G = \exp L$ like $\mathbf{SU}(2) \cong \{e^{i\vec{\sigma}\vec{x}} | \vec{x} \in \mathbb{R}^3\}$ or the symmetric space $\mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \{x \in \mathbb{R}^4 | x^2 = \frac{1}{m^2} \neq 0\}$ by the vectors of a timelike orbit (hyperboloid).

With the dual group $q \in V^T \cong \check{\mathbb{R}}^d$ the representation classes for $G/H$ are characterizable by G-invariants $\{I_1, \ldots, I_r\}$, rational for a compact and rational or continuous for a noncompact Cartan subgroup. The invariants are given by $q$-polynomials and can be built by linear invariants $q = m$ for an abelian group and by quadratic invariants $q^2 = \pm m^2$ for selfdual groups. All energy and momentum invariants will be written in mass units.

Using an appropriate generalized function $\tilde{g}$ on the dual group $V^T \cong \check{\mathbb{R}}^d$ the irreducible **U(1)**-representations $e^{i\langle q,x\rangle}$ of the tangent space Fourier transform $\tilde{g}$ to a matrix element $g$ of the symmetric space representation

$$(G/H)_{\text{repr}} \to \mathbb{C}, \quad x \mapsto g(x) = \int d^d q \tilde{g}(q) e^{iqx}$$

The functions $\tilde{g}$ come as quotient of two polynomials where the invariant zeros of the denominator polynomial $P(q)$ characterize an irreducible representation via a Cartan subgroup representation

$$\tilde{g}(q) \cong \frac{Q(q)}{P(q)} \sim \begin{cases} \frac{1}{q-\mu}, & \mu \in \mathbb{R} \oplus i\mathbb{R}, \quad \text{linear} \\ \frac{(q)^j}{(q^2-m^2)^n}, & m \in \mathbb{R}, \quad \text{compact} \\ \frac{(q)^j}{(q^2-m^2)^n}, & m \in \mathbb{R}, \quad \text{noncompact} \end{cases}$$

$g$ is called a *residual representation* (Saller, 2001a) of $G/H$, the complex rational function $q \mapsto \tilde{g}(q)$ a residual representation function. Many examples are given below.

Residual representations for the tangent space $\log G/H = \log G/\log H$ of a symmetric space $G/H$ will be formulated below.

A representation of a symmetric space $G/H$ contains representations of subspaces $K$, e.g., of subgroups $\mathbf{SO}(2) \subset \mathbf{SO}(3)$ or $\mathbf{SO}_0(1,1) \subset \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$. A residual $G/H$-representation with canonical tangent space parameters $x =$

$(x_K, x_\perp)$ has a *projection* to a residual $K$-representation by integration $\int d^{d-s}x_\perp$ over the complementary space[3] $\frac{\log G/H}{\log K} \cong \mathbb{R}^{d-s}$—in both examples above the two-sphere $\frac{\mathbf{SO}_0(1,3)/\mathbf{SO}(3)}{\mathbf{SO}_0(1,1)} \cong \Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$

$$K \longrightarrow \mathbb{C}, \quad x_K \mapsto g(x_K, 0) = \int \frac{d^{d-s}x_\perp}{(2\pi)^{d-s}} g(x) = \int d^s q_K \tilde{g}(q_K, 0) e^{iq_K x_K}$$

The integration picks up the Fourier components for trivial tangent space forms (momenta) $q_\perp = 0$ of $\frac{\log G/H}{\log K}$. More explicit examples are given below.

The method of residual representations tries to translate the relevant representation structures—invariants, Lie algebras, product representations etc.—into the language of rational complex functions $\mathbb{C} \ni q \mapsto \frac{Q(q)}{P(q)} \in \overline{\mathbb{C}}$ with its poles and its residues.

## 3. RESIDUAL REPRESENTATIONS OF THE REALS

The simplest case of residual representations is realized by time representations with energy functions (distributions) and one-dimensional (1D) position representations with momentum functions (distributions) in the real 1D compact group $\mathbf{U}(1) = \exp i\mathbb{R}$ and noncompact group[4] $\mathbf{D}(1) = \exp \mathbb{R}$ with their selfdual doublings $\mathbf{SO}(2)$ and $\mathbf{SO}_0(1, 1)$, respectively.

### 3.1. Nondecomposable Representations of $\mathbb{R}$

The nondecomposable representations (Boerner, 1955; Saller, 1989) of the noncompact totally ordered group $\mathbb{R}$ are the product of an irreducible factor and a nil-factor

$$\mathbb{R} \ni x \mapsto e^{i(\mu+\mathcal{N}_N)x} \in \mathbf{GL}(\mathbb{C}^{1+N}), \quad N = 0, 1, 2, \ldots$$
$$e^{i\mu x} \in \mathbf{GL}(\mathbb{C}), \quad e^{i\mathcal{N}_N x} \in \mathbf{SL}(\mathbb{C}^{1+N})$$

$N$ is called the *nildimension*. The irreducible 1D representations $x \mapsto e^{i\mu x}$ with $N = 0$ are compact for $\mathbb{R} \to \mathbf{U}(1)$ with real *invariant* $\mu$ or noncompact for $\mathbb{R} \to \mathbf{D}(1)$ with imaginary invariant. The matrix elements of the nil-factor with nilpotent matrix $\mathcal{N}$ involve powers in the Lie parameter up to order $N$, e.g.,

$$\mathcal{N}_3 = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \Rightarrow \mathcal{N}_3^3 \neq 0, \quad \mathcal{N}_3^4 = 0, \quad e^{i\mathcal{N}_3 x} = \begin{pmatrix} 1 & ix & \frac{(ix)^2}{2!} & \frac{(ix)^3}{3!} \\ 0 & 1 & x & \frac{(ix)^2}{2!} \\ 0 & 0 & 1 & ix \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[3] $\log G$ denotes the Lie algebra of the Lie group $G$.

[4] With two symbols for the isomorphic Lie groups $\mathbb{R} \cong \mathbf{D}(1)$, both a multiplicative and additive notation can be used. Therewith, one has different notations for the Lie group $\mathbf{D}(1)$ and its Lie algebra $\mathbb{R} = \log \mathbf{D}(1)$.

The representation space of a nondecomposable $\mathbb{R}$-representations can be spanned by $(1 + N)$ principal vectors wherefrom only one can be chosen as an eigenvector.

The irreducible time or spacetime representations in the quantum probability inducing group $\mathbf{U}(1)$ are used for particles (states) with the eigenvalue $m \in \mathbb{R}$ as energy or mass. Nondecomposable, reducible representations come with indefinite unitary groups which cannot be used for a probability interpretation. Therefore, the principal vectors involved—also the one eigenvector—cannot be used to describe particles in quantum theory (Becchi *et al.*, 1976; Kugo and Ojima, 1978; Saller, 1992a,b).

The product of nondecomposable, reducible representations can contain irreducible ones, e.g.

$$e^{im_1 x}\begin{pmatrix}1 & ix\\ 0 & 1\end{pmatrix}\otimes e^{im_2 x}\begin{pmatrix}1 & ix\\ 0 & 1\end{pmatrix}\cong e^{i(m_1+m_2)x}\left(\begin{array}{c|ccc}1 & 0 & 0 & 0\\ \hline 0 & 1 & ix & \frac{(ix)^2}{2}\\ 0 & 0 & 1 & ix\\ 0 & 0 & 0 & 1\end{array}\right)$$

The order structure of the reals defines the additive cones (monoids) $\mathbb{R}_{\vee,\wedge}$ and the bicone (bimonoid) $\mathbb{R}_\vee \uplus \mathbb{R}_\wedge \cong \mathbb{R}_\vee \mathbb{I}(2)$ which is set-isomorphic to the group $\mathbb{R}$. The bicone representations come with a trivial or faithful representation of the sign $\epsilon(x) = \frac{x}{|x|} \in \mathbb{I}(2) = \{\pm 1\}$, the cone representation matrix elements use Heaviside's step function $\vartheta(\pm x) = \frac{1\pm\epsilon(x)}{2}$

Therewith the $\mathbb{R}$-representation matrix elements are complex linear combinations of the $\mathbb{R}$-functions

$$\vartheta(\pm x)x^N e^{i\mu x} = \left(\frac{\partial}{\partial i\mu}\right)^N \vartheta(\pm x)e^{i\mu x}, \quad N = 0, 1, \ldots, \quad \mu \in \mathbb{R} \oplus i\mathbb{R}$$

The nilpotent powers arise by derivations with respect to the invariant.

## 3.2. Rational Complex Representation Functions

An irreducible $\mathbf{U}(1)$ representation of the group $\mathbb{R}$—formulated in this subsection in an application for time $t \in \mathbb{R}$ and energy—can be written as a *residue of a rational complex energy function* or, equivalently, with a *Dirac distribution* supported by the invariant energy $m \in \mathbb{R}$

$$\mathbb{R} \ni t \mapsto e^{imt} = \oint \frac{dq}{2i\pi}\frac{1}{q-m}e^{iqt} = \int dq\, \delta(q-m)e^{iqt} \in \mathbf{U}(1)$$

$$\mathbb{R} \ni 0 \mapsto 1 = \oint \frac{dq}{2i\pi}\frac{1}{q-m}$$

This gives the prototype of a residual representation. The integral $\oint$ circles the singularity in the mathematically positive direction.

For the group $\mathbf{D}(1) \cong \mathbb{R}$, where the dimension coincides with the rank and where the eigenvalues $q$ are the group invariants $m$, the transition to the residual form is a trivial transcription to the singularity $q = m$. This will be different for groups with dimension strictly larger than rank, e.g., for the space rotations $\mathbf{SO}(3)$, having dimension 3 and rank 1, with the invariant a square $\vec{q}^2 = m^2$ of the three possible eigenvalues $\vec{q}$.

The *Dirac and principal value* P *distributions* from $\mathcal{S}'(\check{\mathbb{R}})$ are the real and imaginary part, respectively, of the *causal* (*advanced and retarded*) *distributions*

$$[N|m]_{\vee,\wedge} = \left(\frac{d}{dm}\right)^N [0|m]_{\vee,\wedge} = \frac{[N|m]_\delta \pm i[N|m]_P}{2}, \quad N = 0, 1, \ldots$$
$$\cong \pm\frac{1}{2i\pi}\frac{\Gamma(1+N)}{(q \mp io - m)^{1+N}} = \frac{1}{2}\left[\delta^{(N)}(m-q) \pm \frac{1}{i\pi}\frac{\Gamma(1+N)}{(qP - m)^{1+N}}\right]$$

In the Fourier transformations to $\mathcal{S}'(\mathbb{R})$ the real–imaginary decomposition goes with the order function decomposition $\vartheta(\pm t) = \frac{1+\epsilon(t)}{2}$ leading to representation matrix elements of future $\mathbb{R}_\vee$ and past $\mathbb{R}_\wedge$, of bicone and group

$$\mathbb{R}_{\vee,\wedge} \ni \vartheta(\pm t)t \mapsto \pm\int \frac{dq}{2i\pi}\frac{\Gamma(1+N)}{(q \mp io - m)^{1+N}}e^{iqt} = \vartheta(\pm t)(it)^N e^{imt}$$
$$\mathbb{R}_\vee \uplus \mathbb{R}_\wedge \ni t \mapsto \int \frac{dq}{i\pi}\frac{\Gamma(1+N)}{(qP - m)^{1+N}}e^{iqt} = \epsilon(t)(it)^N e^{imt}$$
$$\mathbb{R} \ni t \mapsto \oint \frac{dq}{2i\pi}\frac{\Gamma(1+N)}{(q-m)^{1+N}}e^{iqt}$$
$$= \int dq\,\delta^{(N)}(m-q)e^{iqt} = (it)^N e^{imt}$$

All those distributions originate from the representation functions in the closed complex plane (Riemannian sphere) $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ with one pole

$$\overline{\mathbb{C}} \ni q \mapsto \frac{1}{q - m^{1+n}} \in \overline{\mathbb{C}}$$

The position $q = m$ and the order $1 + N$ of the singularity is related to the continuous *invariant* and the *dimensionality* of time representation. A trivial nildimension $N$ belongs to a *simple pole* $\frac{1}{q-m}$. A possibly nontrivial $t^N$-dependence, $N \geq 1$ is expressed by the *multipoles* $\frac{1}{(q-m)^{1+N}}$. The *pole normalization* for the representation is given by the residue at the invariant

$$\operatorname*{Res}_m \sum_{n=0}^{N} \frac{a_{-1-n}}{(q-m)^{1+n}} = \sum_{n=0}^{N} \oint \frac{dq}{2i\pi}\frac{a_{-1-n}}{(q-m)^{1+n}} = a_{-1}$$

The complex functions for $a_{-1} = 1$ are appropriately normalized for the representation of the neutral group element. The Fourier transforms with combinations of different contour directions around the pole represent via $\vartheta(\pm t)$ and $\epsilon(t)$ the causal structure of the reals.

The product $\bullet$ of nondecomposable time representation matrix elements comes with the convolution $*$ of the energy distributions reflecting the order and the real–imaginary structure

| $\bullet$ | $\vartheta(t)$ | $\vartheta(-t)$ | $1$ | $-i\epsilon(t)$ |
|---|---|---|---|---|
| $\vartheta(t)$ | $\vartheta(t)$ | $0$ | $\vartheta(t)$ | $-i\vartheta(t)$ |
| $\vartheta(-t)$ | | $\vartheta(-t)$ | $\vartheta(-t)$ | $i\vartheta(-t)$ |
| $1$ | | | $1$ | $-i\epsilon(t)$ |
| $-i\epsilon(t)$ | | | | -1 |

$\Rightarrow$

| $*$ | $[N_1\|m_1]_\vee$ | $[N_1\|m_1]_\vee$ | $[N_1\|m_1]_\delta$ | $[N_1\|m_1]_{\mathrm{P}}$ |
|---|---|---|---|---|
| $[N_2\|m_2]_\vee$ | $[N_+\|m_+]_\vee$ | $0$ | $[N_+\|m_+]_\vee$ | $-i[N_+\|m_+]_\vee$ |
| $[N_2\|m_2]_\wedge$ | | $[N_+\|m_+]_\wedge$ | $[N_+\|m_+]_\wedge$ | $i[N_+\|m_+]_\wedge$ |
| $[N_2\|m_+]_\delta$ | | | $[N_+\|m_+]_\delta$ | $[N_+\|m_+]_{\mathrm{P}}$ |
| $[N_2\|m_+]_{\mathrm{P}}$ | | | | $-[N_+\|m_+]_\delta$ |

with $\begin{array}{ll} N_+ & = N_1 + N_2 \\ m_+ & = m_1 + m_2 \end{array}$

All these distributions span a unital algebra with conjugation with the Dirac distributions a unital subalgebra. The causal distributions for the representations of the cones $\mathbb{R}_{\vee,\wedge}$ constitute nonunital subalgebras that annihilate each other. The principal value distributions are a vector subspace with the convolutive action of the Dirac distribution subalgebra the Dirac distributions for the group $\mathbb{R}$-representations a unital convolution algebra.

### 3.3. Compact Invariants

Poles at a squared representation invariant $q^2 = m^2$ (compact invariant) can be combined from linear poles at $q = \pm|m|$ the invariants for the dual irreducible subrepresentations involved, formulated in this subsection for time and energy.

In addition to the *causal* (*advanced and retarded*) *energy distributions* $[m^2]_{\vee,\wedge}$ there are the (*anti*-) *Feynman energy-distributions* $[m^2]_\pm$ (different normalization factor $\frac{1}{2}$)

$$[m^2]_{\vee,\wedge} = \frac{[m^2]_\epsilon \pm i[m^2]_{\mathrm{P}}}{2} \quad \cong \pm\frac{1}{2i\pi}\frac{|m|}{(q \mp io)^2 - m^2} = \pm\frac{1}{4i\pi}\left(\frac{1}{q \mp io - |m|} - \frac{1}{q \mp io + |m|}\right)$$

$$[m^2]_\pm = [m^2]_\delta \pm i[m^2]_{\mathrm{P}} \quad \cong \pm\frac{1}{i\pi}\frac{|m|}{q^2 \mp io - m^2} = \pm\frac{1}{2i\pi}\left(\frac{1}{q \mp io - |m|} - \frac{1}{q \pm io + |m|}\right)$$

The principal value distribution as imaginary part is combined with the (anti-) symmetric Dirac distributions as real part

$$\left.\begin{matrix} [m^2]_\epsilon \cong |m|\epsilon(q)\delta(q^2-m^2) \\ [m^2]_\delta \cong |m|\delta(q^2-m^2) \end{matrix}\right\} \quad \text{with} \quad i[m^2]_{\rm P} \cong \frac{1}{i\pi}\frac{|m|}{q_{\rm P}^2-m^2}$$

There arise the Dirac distributions with positive and negative energy support

$$\begin{pmatrix} 1 \\ \epsilon(q) \end{pmatrix}\delta(q^2-m^2) = \delta_\vee(q^2-m^2) \pm \delta_\wedge(q^2-m^2)$$

$$\text{with} \quad \delta_{\vee,\wedge}(q^2-m^2) = \vartheta(\pm q)\delta(q^2-m^2) = \frac{1}{2|m|}\delta(q \mp |m|)$$

The Fourier transforms together with those of

$$\pm\frac{1}{2i\pi}\frac{q}{(q \mp io)^2-m^2}, \quad \pm\frac{1}{i\pi}\frac{q}{q^2 \mp io - m^2}, \quad \text{etc.}$$

are representations of the cones and the group with $\mathbb{I}(2) \times \mathbf{SO}(2)$ matrix elements

$$\begin{aligned}
\text{Casual:}\mathbb{R}_{\vee,\wedge} \ni \vartheta(\pm t)t &\mapsto \pm\int \frac{dq}{2i\pi}\frac{\binom{|m|}{q}}{(q \mp io)^2-m^2}e^{iqt} = \vartheta(\pm t)\begin{pmatrix} i\sin|m|t \\ \cos mt \end{pmatrix} \\
\text{Feynman:}\mathbb{R}_\vee \uplus \mathbb{R}_\wedge \ni t &\mapsto \pm\int \frac{dq}{i\pi}\frac{\binom{|m|}{q}}{q^2 \mp io - m^2}e^{iqt} = \begin{pmatrix} 1 \\ \pm\epsilon(t) \end{pmatrix}e^{\pm i|mt|} \\
\text{Bicone:}\mathbb{R}_\vee \uplus \mathbb{R}_\wedge \ni t &\mapsto \int \frac{dq}{i\pi}\frac{\binom{|m|}{q}}{q_{\rm P}^2-m^2}e^{iqt} = \epsilon(t)\begin{pmatrix} i\sin|m|t \\ \cos mt \end{pmatrix} \\
\text{Group:}\mathbb{R} \ni t &\mapsto \int dq\begin{pmatrix} |m| \\ q \end{pmatrix}\epsilon(q)\delta(q^2-m^2)e^{iqt} = \begin{pmatrix} i\sin|m|t \\ \cos mt \end{pmatrix} \\
\text{Group:}\mathbb{R} \ni t &\mapsto \int dq\begin{pmatrix} |m| \\ q \end{pmatrix}\delta(q^2-m^2)e^{iqt} = \begin{pmatrix} \cos mt \\ i\sin|m|t \end{pmatrix}
\end{aligned}$$

By derivation with respect to the invariant there arise distributions with nontrivial nildimensions $\frac{1}{(q^2-m^2)^{1+N}}$.

The convolution properties can be read off the time function multiplication. The Feynman energy-distributions combine real–imaginary and order properties of time $t$ and energies $m^2$ as follows

| $*$ | $\left[m_1^2\right]_+$ | $\left[m_1^2\right]_+$ |
|---|---|---|
| $\left[m_2^2\right]_+$ | $[m_+^2]_+$ | $\vartheta\left(m_1^2-m_2^2\right)[m_-^2]_-$<br>$\vartheta\left(m_2^2-m_1^2\right)[m_-^2]_+$ |
| $\left[m_2^2\right]_-$ | | $[m_+^2]_-$ |

with $\quad m_\pm = |m_1| \pm m_2$

The Feynman distributions $[m^2]_\pm$ for the bicone representations form unital subalgebras. In contrast to the advanced and retarded distributions $[m^2]_{\vee,\wedge}$ they do not annihilate each other.

## 3.4. Noncompact Invariants

The functions with imaginary poles from a negative squared representation invariant $q^2 = -m^2$ (noncompact invariant)

$$[-m^2] \cong \frac{1}{\pi}\frac{|m|}{q^2 - m^2}$$

give, by their Fourier transforms, bicone representations with noncompact $\mathbf{D}(1)$ matrix elements, valued in the convolution algebra[5] $L^1(\mathbb{R})$ and the Hilbert space $L^2(\mathbb{R})$—formulated in this subsection for 1D position $z \in \mathbb{R}$ and momentum $q \in \check{\mathbb{R}}$

$$\mathbb{R}_\vee \times \mathbb{I}(2) \ni z \mapsto \begin{cases} \int \frac{dq}{\pi}\frac{|m|}{q^2+m^2}e^{-iqz} & = \oint \frac{dq}{2i\pi}\left[\frac{\vartheta(-z)}{q-i|m|}\right] - \frac{\vartheta(z)}{q+i|m|}e^{-iqz} = e^{-|mz|} \\ \int \frac{dq}{\pi}\frac{iq}{q^2+m^2}e^{-iqz} & = \epsilon(z)e^{-|mz|} \\ \int \frac{dq}{\pi}\frac{2m^2iq}{(q^2+m^2)^2}e^{-iqz} & = |m|ze^{-|mz|} \end{cases}$$

The representation relevant residues are taken at imaginary "momenta" $q = \pm i|m|$ in the complex momentum plane.

The momentum functions constitute a real unital convolution algebra

$$\left[-m_1^2\right] * \left[-m_2^2\right] = [-m_+^2]$$

The residues of the complex representation functions for compact (real) and noncompact (imaginary) invariant $\mu \in \{\pm|m|, \pm i|m|\}$ are

$$\mu \in \mathbb{C} : \underset{\mu}{\text{Res}} \frac{2}{q^2-\mu^2} = \frac{1}{\mu}, \quad \underset{\mu}{\text{Res}} \frac{2q}{q^2-\mu^2} = 1$$

Higher order pole residues are obtained by $\mu^2$-derivations.

[5] The convolution algebra $L^1(G)$ of a Lie group coincides, for a finite group, with the group algebra $\mathbb{C}^G$.

The residual normalization for the unit element of the group, possible for compact and noncompact invariant, is different from a Hilbert space normalization, possible for a noncompact invariant only, e.g. $\int_{-\infty}^{\infty} dz e^{-|mz|} = \frac{2}{|m|}$.

## 4. RESIDUAL REPRESENTATIONS OF THREE-DIMENSIONAL POSITION (FREE PARTICLES AND BOUND WAVES)

Position representations with compact invariants $\vec{q}^2 = m^2$ (real momenta) are used for wave functions of quantum mechanical free scattering states (free particles) whereas those with noncompact invariants $\vec{q}^2 = -m^2$ (imaginary "momenta") arise in quantum mechanical bound waves.

The representations of 1D position with compact and noncompact invariants can be embedded[6] into rotation $\mathbf{SO}(3)$ compatible representations of three-dimensional (3D) position with the radial position $|z| \cong |\vec{x}| = r \in \mathbb{R}_\vee$ and the compact 2-sphere $\Omega^2$ that extends the sign $\mathbb{I}(2)$ for the two hemispheres

$$\mathbb{R} = \mathbb{R}_\vee \times \mathbb{I}(2) \ni z \hookrightarrow \vec{x} \ni \mathbb{R}^3 \cong \mathbb{R}_\vee \times \Omega^2$$

$$\mathbb{I}(2) \ni \epsilon(z) = \frac{z}{|z|} \hookrightarrow \frac{\vec{x}}{r} \in \Omega^2, \quad r \neq 0$$

In the Pauli representation for position translations by traceless hermitian complex $2 \times 2$ matrices

$$\vec{x} = \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix} = x_a \sigma_a \in \mathbb{R}^3$$

the polar decomposition looks as follows with $u \in \mathbf{SU}(2)$ for the 2-sphere $\Omega^2 \cong \mathbf{SU}(2)/\mathbf{SO}(2)$

$$\vec{x} = u\left(\frac{\vec{x}}{r}\right) \circ \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \circ u^\star\left(\frac{\vec{x}}{r}\right)$$

$$u\left(\frac{\vec{x}}{r}\right) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{2r(r+x_3)}} \begin{pmatrix} r + x_3 & -x_1 + i x_2 \\ -x_1 + i x_2 & r + x_3 \end{pmatrix} \in \mathbf{SU}(2)$$

The Fourier transformations in 3D position are related to those in one dimension by a radial derivative that produces the Kepler factor $\frac{1}{r}$

$$\int \frac{d^3q}{4\pi} \tilde{\mu}(\vec{q}^2) e^{-i\vec{q}\vec{x}} = -\frac{d}{dr^2} \int dq \tilde{\mu}(q^2) e^{-iqr}, \quad \vec{\partial} = \frac{\vec{x}}{r}\frac{d}{dr} = 2\vec{x}\frac{d^2}{dr^2}$$

The integral over the hemisphere directed momentum modulus $q_3 = \epsilon(q_3)|\vec{q}|$ goes over all reals $\int_{-\infty}^{\infty}$. Therewith a function of 1D position $\mathbb{R} \ni z \mapsto f(|z|)$ gives a function of 3D position $\mathbb{R}^3 \ni \vec{x} \mapsto -\frac{d}{dr^2} f(r)$, in the following called 2-sphere spread.

[6] The embedding symbol $\hookrightarrow$ is not meant to imply a unique embedding.

The scalar 3D position representations, nontrivial for $m \neq 0$, use the Fourier transforms with $\mathbf{U}(1)$ and $\mathbf{D}(1)$ matrix elements. For simple poles there arise spherical waves for real momentum poles $|q| = \pm|m|$ and Yukawa potentials for imaginary "momentum" poles $|q| = \pm i|m|$

$$\int \frac{d^3q}{2\pi^2} \frac{1}{\vec{q}^2 \mp io - m^2} e^{-i\vec{q}\vec{x}} = \frac{e^{\pm i|m|r}}{r}, \quad \int \frac{d^3q}{\pi^2} \frac{\mp i|m|}{(\vec{q}^2 \mp io - m^2)^2} e^{-i\vec{q}\vec{x}} = e^{\pm i|m|r}$$

$$\int \frac{d^3q}{2\pi^2} \frac{1}{\vec{q}^2 + m^2} e^{-i\vec{q}\vec{x}} = \frac{e^{-|m|r}}{r}, \quad \int \frac{d^3q}{\pi^2} \frac{|m|}{(\vec{q}^2 + m^2)^2} e^{-i\vec{q}\vec{x}} = e^{-|m|r}$$

which are the 2-sphere spreads of the representations of 1D position $\mathbb{R}$

$$\frac{e^{-\mu r}}{r} = \frac{2}{\mu} \frac{d}{dr^2} e^{-\mu r}, \quad \mu \in \{\mp i(|m| \pm io), |m|\}$$

Position derivations produce momentum polynomials in the numerator for nontrivial 2-sphere representations

$$\int \frac{d^3q}{2\pi^2} \frac{i\vec{q}}{\vec{q}^2 + \mu^2} e^{-i\vec{q}\vec{x}} = -\vec{\partial} \frac{e^{-\mu r}}{r} = \frac{\vec{x}}{r} \frac{1 + \mu r}{r^2} e^{-\mu r},$$

$$\int \frac{d^3q}{\pi^2} \frac{i\vec{q}}{(\vec{q}^2 + \mu^2)^2} e^{-i\vec{q}\vec{x}} = \vec{\partial} \frac{e^{-\mu r}}{\mu} = \frac{\vec{x}}{r} e^{-\mu r}$$

e.g., the Yukawa force for a noncompact invariant $\mu = |m|$.

Nontrivial 2-sphere properties are represented with spherical harmonics $\left(\frac{\vec{x}}{r}\right)^L = \{Y^L_{L_3}(\varphi, \theta) | L_3 = -L, \ldots, L\}$, e.g., $Y^2(\varphi, \theta) \cong \left(\frac{\vec{x}}{r}\right)^2 = \frac{\vec{x} \otimes \vec{x}}{r^2} = \frac{1}{3}\mathbf{1}_3$. To avoid the $r = 0$ ambiguity they have to be multiplied with appropriate radial powers leading to the harmonic polynomials

$$(\vec{x})^L_{L_3} = r^L Y^L_{L_3}(\varphi, \theta), \quad \begin{cases} \vec{\partial}^2 Y^L_{L_3}(\varphi, \theta) = \frac{L(1+L)}{r^2} Y^L_{L_3}(\varphi, \theta) \\ \vec{\partial}(\vec{x})^L_{L_3} \qquad = 0 \end{cases}$$

The harmonic polynomials have trivial translation properties.

The scalar contributions in position representations come with Bessel functions of half-integer order—the hyperbolic Macdonald functions $k_L$ for noncompact invariants and the spherical Hankel $h^{\pm}_L$, Neumann $n_L$ and Bessel $j_L$ functions for compact invariants. They have angular momentum $L$-independent large distance behavior $(k_L(R), h^{\pm}_L(R)) \overset{R\to\infty}{\to} (\frac{e^{-R}}{R}, \frac{e^{\pm iR}}{R})$ and $L$-dependent small distance behavior

$$k_L(R) = (-R)^L \left(\frac{1}{R}\frac{\partial}{\partial R}\right)^L \frac{e^{-R}}{R} = \frac{1}{R^1 + L} \frac{e^{-R}}{2^L} \sum_{n=0}^{L} \frac{(2L-n)!}{(L-n)!} \frac{(2R)^n}{n!}$$

$$= (\pm i)^{1+L} h^{\pm}(\pm iR) = \frac{e^{-R}}{R}, \quad \frac{1+R}{R^2} e^{-R}, \ldots \quad \overset{R\to 0}{\to} \frac{1}{R^1 + L}$$

$$h_L^{\pm}(R) = n_L(R) \pm i j_L(R) = \frac{e^{\pm iR}}{R}, \quad \frac{1 \mp iR}{R^2} e^{\pm iR}, \ldots$$
$$n_L(R) = \frac{\cos R}{R}, \quad \frac{\cos R + R \sin R}{R^2}, \ldots \quad \overset{R\to 0}{\to} \frac{1}{R^1 + L}$$
$$j_L(R) = \frac{\sin R}{R}, \quad \frac{\sin R + R \cos R}{R^2}, \ldots \quad \overset{R\to 0}{\to} R^L$$

To obtain residual representations, which are defined for $r = 0$, i.e., without ambiguity or even singularity, the momentum degree of the numerator $\frac{1}{(\vec{q}^2)^{N(L)}}$ and the degree of the nominator polynomial $(\vec{q})^{(L)}$ have to leave a nonnegative nildimension $N$ for spin $J = \frac{L}{2}$ representations.

Therewith one obtains for the position $\mathbb{R}^3$-representation matrix elements the Dirac momentum distributions with compact invariant for spin $J$ and nildimension $N$

$$(\vec{q})^{2J} \delta^{(N)}(m^2 - \vec{q}^2) \quad \text{for} \quad \begin{cases} J &= 0, \frac{1}{2}, 1, \ldots \\ N &= 0, 1, 2, \ldots \end{cases}$$

The Fourier transformed Dirac momentum distributions starting from the simple compact representations

$$\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3q}{2\pi} \delta(\vec{q}^2 - m^2) e^{-i\vec{q}\vec{x}} = \frac{\sin |m| r}{r} = |m| j_0(|m|r)$$
$$\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3q}{2\pi} i\vec{q} \delta(\vec{q}^2 - m^2) e^{-i\vec{q}\vec{x}} = \frac{\vec{x}}{r} \frac{\sin |m| r - |m| r \cos mr}{r^2}$$
$$= |m|^3 \vec{x} \frac{j_1(|m|r)}{|m|r}$$

describe free states (free particles). They involve spherical Bessel functions multiplied with appropriate radial powers to yield a regular $r \to 0$ behavior

$$\sqrt{\frac{2}{\pi}} \frac{j_L(R)}{R^L} = \frac{\mathcal{J}_{\frac{1+2L}{2}}(R)}{R^{\frac{1+2L}{2}}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{R^2}{4}\right)^n}{\Gamma\left(\frac{3}{2} + L + n\right) n!}$$

In the *dipoles* with compact invariants

$$\int \frac{d^3q}{\pi} \frac{|m| \pm \vec{q}}{(\vec{q}^2 \mp io - m^2)} e^{-i\vec{q}\vec{x}} = \left(1_2 - \frac{\vec{x}}{r}\right) e^{\mp i|m|\vec{x}}$$

the Dirac distribution derivatives give the representations of the compact group $\mathbf{SU}(2) \cong \exp(i\mathbb{R})^3$ with the group functions valued in the Hilbert space

$L^2(\mathbf{SU}(2)) \subset L^1(\mathbf{SU}(2))$ as subspace[7] of the convolution algebra

$$\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3q}{\pi}(|m| + \vec{q})\delta^t(m^2 - \vec{q}^2)e^{-i\vec{q}\vec{x}}$$
$$= \cos mr - i\frac{\vec{x}}{r} sin|m|r = e^{-i|m|\vec{x}} \in \mathbf{SU}(2)$$

The representation matrix elements from the principal value pole for a compact and noncompact invariant require a sufficiently high order pole

$$\left.\begin{array}{ll} \frac{(\vec{q})^{2j}}{\left(\vec{q}_P^2 \mp m^2\right)^{2+J+N}} & \text{for} \quad J = 0, 1, \ldots \\ \frac{(\vec{q})^{2j}}{\left(\vec{q}_P^2 \mp m^2\right)^{\frac{5}{2}+J+N}} & \text{for} \quad J = \frac{1}{2}, \frac{3}{2}, \ldots \end{array}\right\} \quad \text{and} \quad N = 0, 1, \ldots$$

They start with *dipoles* for the scalars, as to be expected from the additional $\vec{q}^2$-power in the Lebesque measure $d^3q = d\Omega^2\vec{q}^2 d|\vec{q}|$, and with *tripoles* for the vectors

$$\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3q}{\pi^2} \frac{|m|}{\left(\vec{q}_P^2 \mp m^2\right)^2} e^{-i\vec{q}\vec{x}} = (\sin|m|r, e^{-|m|r})$$
$$\int \frac{d^3q}{\pi^2} \frac{i\vec{q}}{\left(\vec{q}_P^2 \mp m^2\right)^2} e^{-i\vec{q}\vec{x}} = \frac{\vec{x}}{r}(-\cos mr, e^{-|m|r})$$
$$\mathbb{R}^3 \ni \vec{x} \mapsto \int \frac{d^3q}{\pi^2} \frac{4m^2 i\vec{q}}{\left(\vec{q}_P^2 \mp m^2\right)^3} e^{-i\vec{q}\vec{x}} = |m|\vec{x}(\sin|m|r, e^{-|m|r})$$

The dipole for the vector is ambiguous for $r = 0$. Representation matrix elements for nontrivial nildimension arise by derivatives $(\frac{\partial}{\partial|m|})^N$ producing higher order poles and additional radial powers $r^N$.

For noncompact invariant the Fourier transforms are valued in the position Hilbert space $L^2(\mathbb{R}^3)$ and in the convolution algebra $L^1(\mathbb{R}^3)$. The scalar dipoles and the vector tripoles, etc., are position representations by Schrödinger functions

$$|1, \vec{0}\rangle \sim \qquad e^{-|m|r} = \int \frac{d^3q}{\pi^2} \frac{|m|}{\left(\vec{q}^2+m^2\right)^2} e^{-i\vec{q}\vec{x}}, \quad |m| = 1$$
$$|2, \vec{1}\rangle \sim 2|m|\vec{x}e^{-|m|r} = \int \frac{d^3q}{\pi^2} \frac{8m^2 i\vec{q}}{\left(\vec{q}^2+m^2\right)^3} e^{-i\vec{q}\vec{x}}, \quad |m| = \frac{1}{2}$$

They arise as knotless waves $|k, \vec{L}\rangle$ of the nonrelativistic hydrogen atom (Messiah, 1965) with angular momentum $\vec{L} \cong (L, L_3)$ and principal quantum number $k$—the inverse of the quantized "imaginary" momentum $|\vec{q}| = \pm i|m|$ as invariant for

[7] For compact spaces one has $L^p(T) \subseteq L^q(T)$ for $p \geq q$.

the position representation

$$|k, \vec{L}\rangle \sim (2|m|\vec{x})^L L^N_{1+2L}(2|m|r)e^{-|m|r}$$

$$E = \frac{\vec{q}^2}{2} = -\frac{m^2}{2}, \quad |m| = \frac{1}{k}, \quad k = 1 + L + N$$

The degree of the Laguerre polynomials

$$L^N_\lambda(\rho) = \left(\rho^{-\lambda} e^\rho \frac{d}{d\rho} \rho^\lambda e^{-\rho}\right)^N \frac{\rho^N}{N!}$$

$$= \sum_{n=0}^{N} \binom{\lambda + N}{\lambda + n} \frac{(-\rho)^n}{n!}, \quad \begin{cases} \mathbb{R} \ni \lambda \neq -1, -2, \ldots \\ N = \deg \mathrm{L}^N_\lambda \end{cases}$$

is the radial quantum number (knot number). Nontrivial knots, i.e., nildimensions $N = 1, 2, \ldots$are obtained by operating with the Laguerre polynomials $r^N e^{-|m|r} = \left(\frac{d}{d|m|}\right)^N e^{-|m|r}$, e.g., for one knot

$$|2, \vec{0}\rangle \sim \quad (2 - 2|m|r)e^{-|m|r} = \quad \int \frac{d^3q}{\pi^2} \frac{4|m|\left(\vec{q}^2 - m^2\right)}{\left(\vec{q}^2 + m^2\right)^3} e^{-i\vec{q}\vec{x}}, \quad |m| = \tfrac{1}{2}$$

$$|3, \vec{1}\rangle \sim 2|m|\vec{x}(4 - 2|m|r)e^{-|m|r} = \int \frac{d^3q}{\pi^2} \frac{48m^2 i\vec{q}\left(\vec{q}^2 + m^2\right)}{\left(\vec{q}^2 + m^2\right)^4} e^{-i\vec{q}\vec{x}}, \quad |m| = \tfrac{1}{3}$$

The convolutions can be read off from the matrix elements, e.g.

$$\frac{1}{\pi^2} \frac{|m_1|}{(\vec{q}^2 + m_1^2)^2} * \frac{1}{\pi^2} \frac{|m_2|\vec{q}}{\left(\vec{q} + m_2^2\right)^3} = \frac{1}{\pi^2} \frac{|m_+|\vec{q}}{\left(\vec{q} + m_+^2\right)^3}, \quad |m_+| = |m_1| + |m_2|$$

The residues of the scalar complex representation functions with the complexified radial degree of freedom, e.g.

$$\mathbb{R} \times \Omega^2 \hookrightarrow \overline{\mathbb{C}} \times \Omega^2 \ni \vec{q} = |\vec{q}| \frac{\vec{q}}{|\vec{q}|} \mapsto \frac{1}{\vec{q}^2 - \mu^2} \in \overline{\mathbb{C}}$$

have to take into account the 2-sphere degrees of freedom

$$\mu \in \mathbb{C} : \operatorname*{Res}_\mu \frac{4\mu}{\left(\vec{q}^2 - \mu^2\right)^n} = \oint_\mu \frac{d^3q}{2i\pi^2} \frac{4\mu}{\left(\vec{q}^2 - \mu^2\right)^n} = \oint_\mu \frac{dq}{2i\pi} \frac{4\mu q^2}{\left(\vec{q}^2 - \mu^2\right)^n}$$

$$= \begin{cases} 2\mu^2, & n = 1 \\ 1, & n = 2 \end{cases}$$

The additional normalization factor $\frac{1}{\pi}$ is discussed in the next section.

The residual normalization is used for the representation of the unit in a Cartan subgroup, e.g., $\mathbf{SO}(2) \subset \mathbf{SO}(3)$ or $\mathbf{SO}_0(1, 1) \subset \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$. It is different from a quadratic form normalization, e.g., with the invariant bilinear Killing form of the **SO**(3)-Lie algebra.

## 5. RESIDUAL NORMALIZATIONS

For the characters $\check{\mathbb{R}}^d$ of the translations $\mathbb{R}^d$ with a signature $(d-s, s)$ metric one has the residual normalizations (Gel' fand and Shilov, 1958) for positive and negative invariants $\mu^2$ (where the $\Gamma$-functions are defined)

$$\begin{array}{l}\mathbf{O}(d-s,s)\vec{\times}\check{\mathbb{R}}^d:\\ \mu^2,\nu\in\mathbb{R}\end{array}\left\{\begin{array}{l}\int \frac{d^dq}{(\pm i)^s\sqrt{\pi^d}}\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)}{(q^2\mp io-\mu^2)^{\frac{d}{2}+1+\nu}}=\frac{2}{d}\int\frac{d^dq}{(\pm i)^s\sqrt{\pi^d}}\frac{q^2\Gamma\left(\frac{d}{2}+2+\nu\right)}{(q^2\mp io-\mu^2)^{\frac{d}{2}+2+\nu}}\\ =\frac{\Gamma(1+\nu)}{(\mp io-\mu^2)^{i+\nu}}=\left\{\begin{array}{ll}\frac{\Gamma(1+\nu)}{\left(-\mu_P^2\right)^{1+\nu}}\pm i\pi\delta^{(\nu)}(\mu^2), & \nu=0,1,2,\ldots\\ \frac{\Gamma(1+\nu)\left[\vartheta(-\mu^2)+e^{\pm i\pi(1+\nu)}\vartheta(\mu^2)\right]}{|\mu^2|^{1+\nu}}, & \nu\neq 0,1,2,\ldots\end{array}\right.\end{array}\right.$$

with the relevant examples for definite signatures (energy and momenta) and indefinite ones for Minkowski energy–momenta $\check{\mathbb{R}}^{1+s}$

$$\left.\begin{array}{r}\check{\mathbb{R}}^1:\pm\int\frac{dq}{i\pi}\frac{1}{q^2\mp io-\mu^2}\\ \mathbf{O}(3)\vec{\times}\check{\mathbb{R}}^3:\pm\int\frac{d^3q}{i\pi^2}\frac{1}{(\vec{q}^2\mp io-\mu^2)^2}\end{array}\right\}=\frac{\vartheta(\mu^2)\mp i\vartheta(-\mu^2)}{|\mu|}$$

$$\left.\begin{array}{r}\mathbf{O}(1,1)\vec{\times}\check{\mathbb{R}}^2:\pm\int\frac{d^2q}{i\pi}\frac{1}{\left(q_0^2-q_3^2\mp io-\mu^2\right)^2}\\ \mathbf{O}(1,3)\vec{\times}\check{\mathbb{R}}^4:\mp\int\frac{d^4q}{i\pi^2}\frac{1}{\left(q_0^2-\vec{q}^2\mp io-\mu^2\right)^3}\end{array}\right\}=\frac{1}{\mu_P^2}\pm i\pi\delta(\mu^2)$$

This shows the addional normalization factor $\pm\frac{1}{\pi}$ if the residues of position $\mathbb{R}$ are embedded into position $\mathbb{R}^3\cong\mathbb{R}_V\times\Omega^2$.

## 6. RESIDUAL REPRESENTATIONS OF TWO-DIMENSIONAL SPACETIME

Residual time and position representations can be embedded into Minkowski spacetime representations. They employ energy–momentum distributions whose Lorentz invariant singularities determine the embedded representations of both time and position.

Two-dimensional (2D) Minkowski spacetime in a diagonal $(2\times 2)$-matrix representation

$$x=\begin{pmatrix}x_0+x_3 & 0\\ 0 & x_0-x_3\end{pmatrix}=x_0\mathbf{1}_2+x_3\sigma_3\in\mathbb{R}^2\cong(\mathbb{R}_\vee\uplus\mathbb{R}_\wedge)\oplus(\mathbb{I}(2)\times\mathbb{R}_\vee)$$

is acted upon with the orthochronous Lorentz group (dual dilatations)

$$\mathbf{SO}_0(1,1): x_0\pm x_3\mapsto e^{\pm\psi_3}(x_0\pm x_3)$$

(without rotation degrees of freedom). It is the noncompact abelian substructure of the Lorentz group $\mathbf{SO}_0(1,3)$ of four-dimensional (4D) spacetime $\mathbb{R}^4$.

## 6.1. Energy–Momentum Distributions

The scalar energy–momentum distributions—(anti-) Feynman and causal (advanced, retarded)—are distinguished by their energy $q_0$ behavior. They are combinations of the (anti-)symmetric Dirac distribution with the principal value distribution

$$\text{Feynman:} \pm \frac{1}{i\pi}\frac{1}{q^2 \mp io - m^2} = \delta(q^2-m^2) \pm \frac{1}{i\pi}\frac{1}{q_P^2 - m^2}$$

$$\text{Causal:} \pm \frac{1}{2i\pi}\frac{1}{(q^2 \mp io)^2 - m^2} = \frac{1}{2}\left[\epsilon(q_0)\delta(q^2-m^2) \pm \frac{1}{i\pi}\frac{1}{q_P^2 - m^2}\right]$$

$$\text{with} \quad (q \mp io)^2 = (q_0 \mp io)^2 - q_3^2$$

$$\begin{pmatrix} 1 \\ \epsilon(q_0) \end{pmatrix} \delta(q^2-m^2) = \delta_\vee(q^2-m^2) \pm \delta_\wedge(q^2-m^2)$$

Multipoles arise by derivations with respect to the invariant $m^2$.

The Fourier transformed $d^2q = dq_0 dq_3$ Dirac distribution for energy–momenta

$$\int d^2q\,\delta(q^2-m^2)e^{iqx} = -\pi\mathcal{N}_0\left(\sqrt{\frac{m^2x^2}{4}}\right)$$

$$= \vartheta(x^2)\pi\mathcal{N}_0\left(\sqrt{\frac{m^2x^2}{4}}\right) + \vartheta(-x^2)2\mathcal{K}_0\left(\sqrt{-\frac{m^2x^2}{4}}\right)$$

comes with the order 0 Neumann function for real argument (timelike) which is the Macdonald function for imaginary argument (spacelike)

$$\mathbb{R} \ni \xi \mapsto \pi\mathcal{N}_0(\xi) = \sum_{n=0}^{\infty} \frac{\left(-\frac{\xi^2}{4}\right)^n}{(n!)^2}\left[\log\frac{|\xi^2|}{4} + 2\gamma_0 - 2\varphi(n)\right] = -2\mathcal{K}_0(-i\xi)$$

$$\varphi(0) = 0, \quad \varphi(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad n = 1, 2, \ldots$$

$$\gamma_0 = -\Gamma'(1) = \lim_{n\to\infty}[\varphi(n) - \log n] = 0.5772\ldots$$

The advanced and retarded Fourier transforms are causally supported

$$\int \frac{d^2q}{i\pi}\frac{1}{q_P^2 - m^2}e^{iqx} = \epsilon(x_0)\int d^2q\,\epsilon(q_0)\delta(q^2-m^2)e^{iqx}$$

$$= i\pi\vartheta(x^2)\mathcal{E}_0\left(\frac{m^2x^2}{4}\right)$$

They involve Bessel functions of integer order

$$\mathbb{R} \ni \xi \mapsto \mathcal{E}_L\left(\frac{\xi^2}{4}\right) = \frac{\mathcal{J}_L(\xi)}{\left(\frac{\xi}{2}\right)^L} = \sum_{n=0}^{\infty} \frac{\left(-\frac{\xi^2}{4}\right)^n}{(L+n)!n!}$$

$$= \left(-\frac{\partial}{\partial \frac{\xi^2}{4}}\right)^L \mathcal{E}_0\left(\frac{\xi^2}{4}\right),$$

$$L = 0, 1, \ldots \mathcal{E}_0\left(\frac{\xi^2}{4}\right) = \mathcal{J}_0(\xi), \quad (1+L)\mathcal{E}_{1+L}\left(\frac{\xi^2}{4}\right)$$

$$= \mathcal{E}_L\left(\frac{\xi^2}{4}\right) + \frac{\xi^2}{4}\mathcal{E}_{2+L}\left(\frac{\xi^2}{4}\right)$$

The Feynman propagators proper—for particles—have first order poles—they come with the Hankel functions $\mathcal{H}_0^{\mp} = \mathcal{N}_0 \mp i\mathcal{J}_0$

$$\pm \int \frac{d^2q}{i\pi} \frac{1}{q^2 \mp io - m^2} e^{iqx} = -\pi\vartheta(x^2)\mathcal{H}_0^{\mp}\left(\sqrt{\frac{m^2x^2}{4}}\right)$$

$$+\vartheta(-x^2)2\mathcal{K}_0\left(\sqrt{-\frac{m^2x^2}{4}}\right)$$

Fourier transformed Lorentz vectors

$$\pm\frac{1}{2i\pi}\frac{q}{(q \mp io)^2 - m^2}, \quad \pm\frac{1}{i\pi}\frac{q}{q^2 \mp io - m^2}, \text{ etc.}$$

are obtained by spacetime derivation $\partial = 2x\frac{\partial}{\partial x^2}$, e.g.

$$\int \frac{d^2q}{i\pi}\frac{q}{q_P^2 - m^2} e^{iqx} = \epsilon(x_0)\int d^2q\, q\epsilon(q_0)\delta(q^2 - m^2)e^{iqx}$$

$$= \pi\partial\vartheta(x^2)\mathcal{E}_0\left(\frac{m^2x^2}{4}\right) = \pi\frac{x}{2}\left[\delta\left(\frac{x^2}{4}\right) - \vartheta(x^2)m^2\mathcal{E}_1\left(\frac{m^2x^2}{4}\right)\right]$$

## 6.2. Time and Position Frames

The partial Fourier transformations with respect to energy and momentum display the spacetime embedded time and position representations

$$\pi g(m^2, x) = \int d^2q\, e^{iqx}\tilde{g}(m^2, q) = \int dq_3\, e^{-iq_3x_3} g(q_0, x_0)$$

$$= \int dq_0\, e^{iq_0x_0}\left[\vartheta\left(q_0^2 - m^2\right)^c(q_3, x_3) + \vartheta\left(m^2 - q_0^2\right)g^{nc}(iq_3, x_3)\right]$$

$$\text{Time: } \mathbb{R} \ni x_0 \mapsto g(q_0, x_0)$$
$$\text{Position: } \mathbb{R} \ni x_3 \mapsto \begin{cases} g(q_3, x_3) \\ g^{nc}(iq_3, x_3) \end{cases}$$

Time and Position Representations for $\mathbb{R}^2$-Spacetime

| | Time (compact) | Position (compact) | Position (noncompact) |
|---|---|---|---|
| $\tilde{g}(m^2, q)$ | $g(q_0, x_0)$ <br> $q_0 = \sqrt{m^2 + q_3^2}$ | $g^c(q_3, x_3)$ <br> $q_3 = \sqrt{q_0^2 - m^2}$ | $g^{nc}(iq_3, x_3)$ <br> $iq_3 = \lvert Q \rvert = \sqrt{m^2 - q_0^2}$ |
| *Lorentz scalars* | | | |
| $\delta(m^2 - q^2)$ | $\frac{\cos q_0 x_0}{q_0}$ | $\frac{\cos q_3 x_3}{q_3}$ | 0 |
| $\epsilon(q_0)\delta(m^2 - q^2)$ | $i \frac{\sin q_0 x_0}{q_0}$ | $\epsilon(q_0) \frac{\cos q_3 x_3}{q_3}$ | 0 |
| $\frac{1}{i\pi} \frac{1}{q_P^2 - m^2}$ | $\epsilon(x_0) i \frac{\sin q_0 x_0}{q_0}$ | $i \frac{\sin q_3 \lvert x_3 \rvert}{q_3}$ | $i \frac{e^{-\lvert Q x_3 \rvert}}{\lvert Q \rvert}$ |
| *Lorentz vectors* | | | |
| $q\delta(m^2 - q^2)$ | $\begin{pmatrix} i \sin q_0 x_0 \\ \frac{q_3}{q_0} \cos q_0 x_0 \end{pmatrix}$ | $\begin{pmatrix} \frac{q_0}{q_3} \cos q_3 x_3 \\ i \sin q_3 x_3 \end{pmatrix}$ | 0 |
| $q\epsilon(q_0)\delta(m^2 - q^2)$ | $\begin{pmatrix} \cos q_0 x_0 \\ \frac{q_3}{q_0} i \sin q_0 x_0 \end{pmatrix}$ | $\epsilon(q_0) \begin{pmatrix} \frac{q_0}{q_3} \cos q_3 x_3 \\ i \sin q_3 x_3 \end{pmatrix}$ | 0 |
| $\frac{1}{i\pi} \frac{q}{q_P^2 - m^2}$ | $\epsilon(x_0) \begin{pmatrix} \cos q_0 x_0 \\ \frac{q_3}{q_0} i \sin q_0 x_0 \end{pmatrix}$ | $- \begin{pmatrix} \frac{q_0}{q_3} i \sin q_3 \lvert x_3 \rvert \\ \epsilon(x_3) \cos q_3 x_3 \end{pmatrix}$ | $\begin{pmatrix} i \frac{q_0}{\lvert Q \rvert} \\ \epsilon(x_3) \end{pmatrix} e^{-\lvert Q x_3 \rvert}$ |

The higher order poles arise by derivation

$$\frac{\partial}{\partial |m|} = 2|m| \frac{\partial}{\partial m^2} \cong \frac{|m|}{q_0} \frac{\partial}{\partial q_0} \cong -\frac{|m|}{q_3} \frac{\partial}{\partial q_3} \cong \frac{|m|}{Q} \frac{\partial}{\partial |Q|}$$

The Dirac distributions involve time and position representations with compact invariant, the principal value part, in addition, also position representations with noncompact invariant $q_3^2 = -(m^2 - q_0^2)$

$$\frac{1}{-q_P^2 + m^2} = \vartheta\left(q_0^2 - m^2\right) \frac{1}{q_3^2 - \left(q_0^2 - m^2\right)} + \vartheta\left(m^2 - q_0^2\right) \frac{1}{q_3^2 + \left(m^2 - q_0^2\right)}$$

The *projection to time representations* will be defined by the partial Fourier transformation $\int dx_3 g(m^2, x)$ leading to trivial momentum $q_3 = 0$ (rest system), defining a *time frame*. The *projection to position representations* by the partial Fourier transformation $\int dx_0 g(m^2, x)$ leads to trivial energy $q_0 = 0$ and defines a *position frame*

$$g(|m|, t) = \int \frac{dx_3}{2} g(m^2, x) = \int d^2q \, \delta(q_3) \tilde{g}(m^2, q) e^{iqx}$$
$$g^{nc}(i|m|, z) = \int \frac{dx_0}{2} g(m^2, x) = \int d^2q \, \delta(q_0) \tilde{g}(m^2, q) e^{iqx}$$

Time frames have real energies for free particles—position frames have "imaginary" momenta for bound waves.

Time and Position Projection for $\mathbb{R}^2 \cong \mathbb{R} \oplus \mathbb{R}$

| | Time frame ($x_0 = t$) | Position frame ($x_3 = z$) |
|---|---|---|
| $\tilde{g}(m^2, q)$ | $g(\lvert m\rvert, t)$ <br> $(q_0, q_3) = (\lvert m\rvert, 0)$ | $g^{nc}(i\lvert m\rvert, z)$ <br> $(q_0, q_3) = (0, i\lvert m\rvert)$ |
| *Lorentz scalars* | | |
| $\delta(m^2 - q^2)$ | $\frac{\cos mt}{\lvert m\rvert}$ | 0 |
| $\epsilon(q_0)\delta(m^2 - q^2)$ | $i\,\frac{\sin mt}{m}$ | 0 |
| $\frac{1}{i\pi}\frac{1}{q_P^2 - m^2}$ | $\epsilon(t)i\,\frac{\sin mt}{m}$ | $i\,\frac{e^{-\lvert mz\rvert}}{\lvert m\rvert}$ |
| *Lorentz vectors* | | |
| $q\delta(m^2 - q^2)$ | $\begin{pmatrix} i\sin\lvert m\rvert t \\ 0 \end{pmatrix}$ | 0 |
| $q\epsilon(q_0)\delta(m^2 - q^2)$ | $\begin{pmatrix} \cos mt \\ 0 \end{pmatrix}$ | 0 |
| $\frac{1}{i\pi}\frac{q}{q_P^2 - m^2}$ | $\begin{pmatrix} \epsilon(t)\cos mt \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ \epsilon(z)e^{-\lvert mz\rvert} \end{pmatrix}$ |
| $\frac{1}{i\pi}\frac{q}{\left(q_P^2 - m^2\right)^2}$ | $\begin{pmatrix} -\frac{t\sin\lvert mt\rvert}{2\lvert m\rvert} \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ -z\frac{e^{-\lvert mz\rvert}}{2\lvert m\rvert} \end{pmatrix}$ |

In the projections there remain the compact time and the noncompact position representations. The Dirac energy–momentum distributions embed only time projections whereas the *principal value distributions embed both time and position* projections. The time representations have nildimensions $N = 0, 1, \ldots$ for poles, dipoles etc. The position projections arise from spacetime distributions with causal support $x^2 \geq 0$.

The complex representation functions for 2D spacetime, e.g.

$$\overline{\mathbb{C}}^2 \ni q \mapsto \frac{q}{q^2 - m^2} \in \overline{\mathbb{C}}^2$$

have energy and momentum projected residues with real and imaginary invariants—for Lorentz scalars

$$\operatorname*{Res}_{\pm i|m|} \frac{2}{q^2 - m^2} = \oint_{\pm|m|} \frac{d^2q}{2i\pi}\delta(q_3)\frac{2}{q^2 - m^2} = \oint_{\pm|m|} \frac{dq}{2i\pi}\frac{2}{q^2 - m^2} = \pm\frac{1}{|m|}$$

$$\operatorname*{Res}_{\pm i|m|} \frac{2}{q^2 - m^2} = \oint_{\pm i|m|} \frac{d^2q}{2i\pi}\delta(q_0)\frac{2}{q^2 - m^2} = -\oint_{\pm i|m|} \frac{dq}{2i\pi}\frac{2}{q^2 + m^2} = -\frac{1}{\pm i|m|}$$

For Lorentz vectors $q = q_0\mathbf{1}_2 + q_3\sigma_3$ with $\operatorname{tr} q = 2q_0$ there is a trace residue for the energy projection

$$\operatorname{tr}\operatorname*{Res}_{\pm i|m|}\frac{q}{q^2-m^2}=\operatorname{tr}\oint_{\pm|m|}\frac{d^2q}{2i\pi}\delta(q_3)\frac{q}{q^2-m^2}=\oint_{\pm|m|}\frac{dq}{2i\pi}\frac{2q}{q^2-m^2}=1$$

Starting from the projections, the compact position representations are induced (Folland, 1995) by the compact time representations with the eigenvalues $(q_0, q_3)$ on the $\mathbf{SO}_0(1,1)$-*mass hyperboloid* and the $\mathbf{SO}_0(1,1)$-measure $\frac{dq^3}{2\sqrt{q_3^2+m^2}}$. The spacetime translation representation has the cardinality of $\mathbf{SO}_0(1,1)$ as its overcountably infinite dimension. The related Dirac distributions for unitary spacetime translation representations embed free scattering waves (free particles)

$$\begin{gathered}(|m|,0)\hookrightarrow(q_0,q_3)\text{ with }e^{i|m|t}\hookrightarrow e^{iq_0x_0-iq_3x_3},\quad q_0^2-q_3^2=m^2\\ \mathbb{R}\ni t\mapsto e^{i|m|t},\quad \mathbb{R}^2\ni x\mapsto\int d^2q\,\delta(q^2-m^2)e^{iqx}\end{gathered}$$

The noncompact position representation matrix elements are functions from the position Hilbert space. They induce time representations with the eigenvalues $(q_0, Q)$ on the SO(2)-*mass circle*. The spacetime embedding for the position bound waves uses the principal value distributions

$$\begin{gathered}(0,i|m|)\hookrightarrow(q_0,iQ)\quad\text{with}\quad e^{-|mz|}\hookrightarrow e^{iq_0x_0-|Qx_3|},\quad q_0^2+Q^2=m^2\\ \mathbb{R}\ni z\mapsto e^{|mz|},\quad\text{causal } x^2\geq 0:\mathbb{R}^2\ni x\mapsto\int d^2q\frac{1}{q_P^2-m^2}e^{iqx}\end{gathered}$$

### 6.3. Singularity Surfaces in Energy–Momenta

For time and 1D position, the representation functions

$$\mathbb{R}\hookrightarrow\overline{\mathbb{C}}\in q\mapsto\frac{1}{q^2\mp m^2}\in\overline{\mathbb{C}}$$

are singular at points in the complex plane $\mathbb{C}\cong\mathbb{R}^2$, at $\{\pm|m|, 0\}$ for compact and at $\{(0, \pm i|m|)\}$ for noncompact representations. For 2-dimensional spacetime, the singularities of

$$\mathbb{R}\oplus\mathbb{R}\hookrightarrow\overline{\mathbb{C}}\oplus\overline{\mathbb{C}}=\overline{\mathbb{C}}^2\ni q\mapsto\frac{1}{q^2-m^2}\in\overline{\mathbb{C}}$$

are on a real 2-dimensional surface in the real 4-dimensional space $\mathbb{C}^2\cong\mathbb{R}^4$ with a complex energy and a complex momentum plane

$$(q_0,\Gamma;q_3,Q)\in\mathbb{R}^4:\begin{cases}q_0^2-\Gamma^2-q_3^2+Q^2=m^2\\ q_0\Gamma-q_3Q=0\end{cases}$$

For nontrivial mass the singularity surface can be parametrized with a positive and negative energy-like hyperboloid and a forward and backward momentum-like hyperboloid

$$m^2 > 0, \quad \mathbf{SO}_0(1,1): \begin{cases} q_0^2 - q_3^2 = m_0^2, & (q_0, q_3) = m_0(\cosh\psi, \sinh\psi) \\ \Gamma^2 - Q^2 = -m_3^2, & (\Gamma, Q) = m_3(\sinh\psi, \cosh\psi) \end{cases}$$

For four spacetime dimensions the momentum-like hyperboloid has one shell only, $\epsilon(z) \hookrightarrow \frac{\vec{x}}{r}$. The singularity surface contains the circles

$$\mathbf{SO}(2): \begin{cases} m_0^2 + m_3^2 = m^2 \Rightarrow (m_0, m_3) = |m|(\cos\alpha, \sin\alpha) \\ q_0^2 + Q^2 = m^2 \cosh^2\psi \\ q_3^2 + \Gamma^2 = m^2 \sinh^2\psi \end{cases}$$

Therewith, the singularity surface in $\mathbb{C}^2$ is four times a circle, embedding the imaginary poles for noncompact $\mathbb{R}$-representations, sliding along a hyperboloid which embeds the real poles for compact $\mathbb{R}$-representations. It can be seen in the $\mathbb{R}^3$-projection to real energies where the energy–momentum hyperbola touches the energy–imaginary "momentum" circle at the two points $(\pm|m|, 0; 0, 0)$

$$\mathbb{R} \oplus \mathbb{C} \ni (q_0, 0; q_3, Q): \quad \left\{q \,|\, q_0^2 - q_3^2 = m^2, Q = 0\right\} \\ \cup \left\{q \,|\, q_0^2 + Q^2 = m^2, q_3 = 0\right\}$$

and in the $\mathbb{R}^3$-projection to real momenta where there is the energy–momentum hyperbola only

$$\mathbb{C} \oplus \mathbb{R} \ni (q_0, \Gamma; q_3, 0): \left\{q \,|\, q_0^2 - q_3^2 = m^2, \Gamma = 0\right\}$$

For trival invariant the circles shrink to points on the hyperbola

$$m^2 = 0; (\Gamma, Q) = 0 \quad \text{or} \quad (q_0, q_3) = 0 \Rightarrow \text{trival } \mathbf{SO}(2)$$

In $\frac{1}{q^2 - m^2}$, $m^2 > 0$, there is only one Lorentz invariant for the real 2D *hyperbolic-spherical singularity surface*. For representations of nonlinear spacetime below two invariants will be introduced—to embed compact representations $e^{imt}$ and noncompact ones $e^{|m|r}$, each kind with an independent invariant.

## 7. CONVOLUTIONS FOR SPACETIME

Feynman integrals as used in perturbation theory involve convolutions of energy–momentum distributions for pointwise products of spacetime distributions. In general they do not make sense since $\mathcal{S}(\check{\mathbb{R}}^d)$ is no convolution algebra.

For energy–momentum convolutions the points on the hyperbolic-spherical singularity surfaces involved are added. The addition of compact with compact and noncompact with noncompact invariants embed products for time and position representations. The characteristically new feature is the addition of compact with noncompact invariants.

### 7.1. Convolution of Two-Dimensional Energy–Momentum Distributions

The product of Feynman propagators for product representations of spacetime uses the convolution of energy–momentum distributions where $\delta(\sum_j q_j - q)$ adds up the energy–momenta as spacetime translation eigenvalues to the eigenvalue $q$ of the product representation, e.g., for scalar multipole Feynman propagators

$$\begin{aligned}
&\pm\frac{1}{i\pi}\frac{\Gamma(1+n_1)}{\left(q^2 \mp io - m_1^2\right)^{1+n_1}} * \cdots * \pm\frac{1}{i\pi}\frac{\Gamma(1+n_k)}{\left(q^2 \mp io - m_k^2\right)^{1+n_k}} \\
&\quad = \left(\pm\frac{1}{i\pi}\right)^k \int d^2q_1 \cdots d^2q_k \delta\left(\sum_{j=1}^k q_j - q\right) \prod_{j=1}^k \frac{\Gamma(1+n_j)}{\left(q_j^2 \mp io - m_j^2\right)^{1+n_j}}
\end{aligned}$$

The convoluted Feynman distributions have to be all of the same type, either all advanced $q^2 - io$ or all retarded $q^2 + io$.

The convolution is performed by joining first the invariant determining quadratic denominator polynomials of the energy–momentum distributions

$$\begin{aligned}
\frac{\Gamma(\nu_1)\cdots\Gamma(\nu_k)}{R_1^{\nu_1}\cdots R_k^{\nu_1}} &= \int_0^1 d\zeta_1 \cdots \int_0^1 d\zeta_k \delta(\zeta_1 + \cdots + \zeta_k - 1) \\
&\quad \times \frac{\zeta_1^{\nu_1 - 1}\cdots\zeta_k^{\nu_{k-1}}\Gamma(\nu_1 + \cdots + \nu_k)}{(R_1\zeta_1 + \cdots + R_k\zeta_k)^{\nu_1+\cdots+\nu_k}} \\
&\quad \nu_j \in \mathbb{R}, \quad \nu_j \neq 0, -1, -2, \ldots
\end{aligned}$$

e.g., for two Feynman distributions

$$\begin{aligned}
&\pm\frac{1}{i\pi}\frac{\binom{1}{q}\Gamma(1+n_1)}{\left(q^2 \mp io - m_1^2\right)^{1+n_1}} * \pm\frac{1}{i\pi}\frac{\binom{1}{q}\Gamma(1+n_2)}{\left(q^2 \mp io - m_2^2\right)^{1+n_2}} \\
&= \frac{1}{i\pi}\int_0^1 d\zeta_{1,2}\delta(\zeta_1+\zeta_2-1)\int\frac{d^2p}{i\pi}\frac{\begin{pmatrix}1 & q\zeta_1 \\ q\zeta_2 & -p\otimes p + q\otimes q\zeta_1\zeta_2\end{pmatrix}\zeta_1^{n_1}\zeta_2^{n_2}\Gamma(2+n_1+n_2)}{\left[p^2 \mp io + q^2\zeta_1\zeta_2 - m_2^2\zeta_2\right]^{2+n_1+n_2}}
\end{aligned}$$

For the integration the tensor $p \otimes p - q \otimes q\zeta_1\zeta_2$ can be replaced by $\mathbf{1}_2\frac{p^2}{2} - q \otimes q\zeta_1\zeta_2$.

The convolution is the *q-dependent residue of the relative energy–momenta* $p = q_1 - q_2$

$$\pm\int\frac{d^2p}{i\pi}\frac{\Gamma(2+n)}{(p^2 \mp io + a)^{2+n}} = \pm\int\frac{d^2p}{i\pi}\frac{p^2\Gamma(3+n)}{(p^2 \mp io + a)^{3+n}} = \frac{\Gamma(1+n)}{(\mp io + a)^{1+n}}$$

which leads to

$$\pm\frac{1}{i\pi}\frac{\Gamma(1+n_1)}{\left(q^2\mp io-m_1^2\right)^{1+n_1}}*\pm\frac{1}{i\pi}\frac{\Gamma(1+n_2)}{\left(q^2\mp io-m_2^2\right)^{1+n_2}}$$

$$=\pm\frac{1}{i\pi}\int_0^1 d\zeta\,\frac{\zeta^{n_1}(1-\zeta)^{n_2}\Gamma(1+n_1+n_2)}{\left[(q^2\mp io)\zeta(1-\zeta)-m_1^2\zeta-m_2^2(1-\zeta)\right]^{1+n_1+n_2}}$$

Here and in the following the convolutions exist only for pole orders where the involved $\Gamma$-functions are defined. Elsewhere, there arise “divergencies.”

### 7.2. Compact and Noncompact Convolution Contributions

The convolution of two Feynman distributions for $s$-dimensional position $\mathbb{R}^s$

$$\pm\int\frac{d^{1+s}q}{i\pi}\frac{1}{q^2\mp io-m^2}e^{iqx}=\int d^{1+s}q[1\pm\epsilon(q_0x_0)]\delta(q^2-m^2)e^{iqx}$$

gives as real part the difference of the squares of Dirac and principal value contributions (with $\epsilon(x_0)^2=1$) whereas the imaginary part contains the mixed terms

$$(\delta^1\pm i\mathbf{P}^1)*(\delta^2\pm i\mathbf{P}^2)=\delta^{1*2}\pm i\mathbf{P}^{1*2},\quad\begin{cases}\delta^{1*2}=\delta^1*\delta^2-\mathbf{P}^1*\mathbf{P}^2\\ \mathbf{P}^{1*2}=\mathbf{P}^1*\mathbf{P}^2+\mathbf{P}^1*\delta^2\end{cases}$$

The product of the order functions in the product of two Feynman propagators

$$[1\pm\epsilon(q_0x_0)][1\pm\epsilon(p_0x_0)]=[1+\epsilon(q_0p_0)]\pm[\epsilon(q_0)+\epsilon(p_0)]\epsilon(x_0)$$
$$=2[\vartheta(q_0)\vartheta(p_0)+\vartheta(-q_0)\vartheta(-p_0)]\pm[\epsilon(q_0)+\epsilon(p_0)]\epsilon(x_0)$$

allows the disentanglement of the convolution

$$\pm\frac{1}{i\pi}\frac{1}{q^2\mp io-m_1^2}*\pm\frac{1}{i\pi}\frac{1}{q^2\mp io-m_2^2}$$
$$=\left[\vartheta(+q_0)\delta\left(q^2-m_1^2\right)*\vartheta(+q_0)\delta\left(q^2-m_2^2\right)\right.$$
$$\left.+\vartheta(-q_0)\delta\left(q^2-m_1^2\right)*\vartheta(+q_0)\delta\left(q^2-m_2^2\right)\right]$$
$$\pm\frac{1}{i\pi}\left[\delta\left(q^2-m_1^2\right)*\frac{1}{q_P^2-m_2^2}+\frac{1}{q_P^2-m_1^2}*\delta\left(q^2-m_2^2\right)\right]$$

The convolution with the singularities for nontrivial position $\mathbb{R}^s$ on $s$-dimensional hyperboloids does not lead s-dimensional hyperboloids $\delta(q^2-m_+^2)$, but to *thresholds* for energy–momenta $q^2=(q_1+q_2)^2\geq m_+^2$

$$\vartheta(\pm q_0)\delta\left(q^2-m_1^2\right)*\vartheta(\pm q_0)\delta\left(q^2-m_2^2\right)\sim\vartheta(\pm q_0)\vartheta(q^2-m_+^2)$$

Here, the energy is enough to produce two free real particles with masses $m_{1,2}$. and momentum $(\vec{q}_1 + \vec{q}_2)^2 \geq 0$

$$\vartheta(\pm q_0)\vartheta(q^2 - m^2) = \vartheta(\pm q_0) \int_0^{\vec{q}^2} \delta\vec{p}^2 \delta\big(q_0^2 - \vec{p}^2 - m^2\big)$$

The convolution of two step functions at masses $m_{1,2}$ gives a step function for the sum mass $m_+ = |m_1| + |m_2|$. The set with all $s+1$D forward (backwards) hyperboloids $\{\{q \succeq |m|\} | m \in \mathbb{R}\}$ is an additive cone

$$\{q \succeq |m_1|\} + \{q \succeq |m_2|\} = \{q \succeq |m_+|\}$$
$$\vartheta(\pm q_0)\vartheta\big(q^2 - m_1^2\big) * \vartheta(\pm q_0)\vartheta\big(q^2 - m_2^2\big) \sim \vartheta(\pm q_0)\vartheta(q^2 - m_+^2)$$

The convolution of compact translation representation matrix elements from the real part of the propagator (free particles) gives corresponding matrix elements for product representations (product of free particles). The positive and negative energy–momentum distributions are convolution algebras, not annihilating each other

$$\delta = \delta_\vee + \delta_\wedge, \quad \delta^1 * \delta^2 = \big(\delta_\vee^1 + \delta_\wedge^1\big) * \big(\delta_\vee^2 + \delta_\wedge^2\big)$$
$$\mathbf{P} \sim i\epsilon(x_0)(\delta_\vee - \delta_\wedge), \quad \mathbf{P}^1 * \mathbf{P}^2 = -\big(\delta_\vee^1 - \delta_\wedge^1\big) * \big(\delta_\vee^2 - \delta_\wedge^2\big)$$
$$\delta^{1*2} = \delta^1 * \delta^2 - \mathbf{P}^1 * \mathbf{P}^2 = 2\big(\delta_\vee^1 * \delta_\vee^2 + \delta_\wedge^1 * \delta_\wedge^2\big) \sim \int_{\mathbb{R}^s} 2\delta^{1+2}$$
$$\text{with} \quad \delta_{\vee,\wedge} \in \mathcal{D}'\big(\check{\mathbb{R}}_{\vee,\wedge}^{1+s}\big)$$

For time and energy, also the principal value part adds up the invariant poles

$$\text{only for time}: \mathbf{P}^{1*2} = \delta^1 * \mathbf{P}^2 + \mathbf{P}^1 * \delta^2 \sim 2i\epsilon(t)\big(\delta_\vee^1 * \delta_\vee^2 - \delta_\wedge^1 * \delta_\wedge^2\big) \sim 2\mathbf{P}^{1+2}$$

The characteristic effect of a convolution of noncompact with compact invariant comes in the principal value part for $s = 1, 3$ position degrees of freedom

$$\begin{array}{rcll} \delta(q^2 - m^2) & \sim & \vartheta(q^2 - m^2) & \\ \frac{1}{q_P^2 - m^2} & \sim & \vartheta(q^2 - m^2) & + \vartheta(-q^2 + m^2) \\ & & \cup & \cup \\ & & \text{compact (free)} + & \text{noncompact} \\ & & e^{imt} & e^{-|mz|} \end{array}$$

The two energy–momentum dependent zeros of the denominator polynomial

$$-\mathbf{P}(\zeta) = q^2\zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta) = -q^2\big[\zeta - \zeta_1(q^2)\big]\big[\zeta - \zeta_2(q^2)\big]$$
$$\zeta_{1,2}(q^2) = \frac{q^2 - m + m_- \pm \sqrt{\Delta(q^2)}}{2q^2} \quad \text{with} \quad \begin{cases} m_\pm = |m_1| \pm |m_2| \\ \Delta(q^2) = (q^2 - m_+^2)(q^2 - m_-^2) \end{cases}$$

are either both real or complex conjugate to each other according to the sign of the discriminant $\Delta(q^2)$. Furthermore, real zeros—in the case of $\Delta(q^2) \geq 0$—are in the integration $\zeta$-interval [0, 1] only for energy–momenta over the threshold $\vartheta(q^2 - m_+^2)$. Therewith, the convolution of scalar propagators for 2-dimensional spacetime reads

$$
\begin{aligned}
\mathbb{R}^2 : \pm \frac{1}{i\pi}\frac{1}{q^2 \mp io - m_1^2} * \pm \frac{1}{i\pi}\frac{1}{q^2 \mp io - m_2^2} &= \pm \frac{1}{i\pi}\int_0^1 d\zeta \frac{1}{(q^2 \mp io)\zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta)} \\
&= \int_0^1 d\zeta \left[\delta\Big(q^2\zeta(1-\zeta) - m_2^2\zeta - m_2^2(1-\zeta)\Big) \pm \frac{1}{i\pi}\frac{1}{q_P^2\zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta}\right] \\
&= \frac{2}{\sqrt{|\Delta(q^2)|}}\Bigg[\vartheta(q^2 - m_+^2) \mp \frac{1}{i\pi}\vartheta(-\Delta(q^2)) \quad \arctan \quad \frac{2\sqrt{-\Delta(q^2)}}{\sum(q^2)} \\
&\quad \mp \frac{1}{i\pi}\vartheta(-\Delta(q^2)) \quad \log\left|\frac{\sum(q^2) - 2\sqrt{\Delta(q^2)}}{m_+^2 - m_-^2}\right|\Bigg] \\
&\quad \text{with} \quad \sum(q^2 - m_+^2) + (q^2 - m_+^2)
\end{aligned}
$$

The spacetime original convolution of compact with noncompact invariants is proportional to $\vartheta(-q^2 - m_-^2)$ and comes in the logarithm

$$
\begin{aligned}
\Delta(-\Delta(q^2)) &= -\vartheta(q^2 - m_+^2) + \vartheta(q^2 - m_-^2) \\
\Delta(\Delta(q^2)) &= \vartheta(q^2 - m_+^2) + \vartheta(m_-^2 - q^2) \\
\left|\frac{\sum(q^2) - 2\sqrt{\Delta(q^2)}}{m_+^2 - m_-^2}\right| &= \left|\frac{\left(\sqrt{m_+^2 - q^2} - \sqrt{m_-^2 - q^2}\right)^2}{m_+^2 - m_-^2}\right|
\end{aligned}
$$

In the correspondingly computed convolution of energy distributions the integral compensates the $m_-^2$-pole from the discriminant

$$
\begin{aligned}
\mathbb{R} : \pm \frac{1}{i\pi}\frac{|m_1|}{q^2 \mp io - m_1^2} * \pm \frac{1}{i\pi}\frac{|m_2|}{q^2 \mp io - m_2^2} \\
= \pm \frac{1}{i\pi}\int_0^1 d\zeta \frac{|m_1 m_2|}{\big[-(q^2 \mp io)\zeta(1-\zeta) + m_1^2\zeta - m_2^2(1-\zeta)\big]^{\frac{3}{2}}} \\
= \pm \frac{1}{i\pi}\frac{|m_+|}{q^2 \mp io - m_+^2}
\end{aligned}
$$

$$\text{with} \quad \frac{1}{P(\zeta)^{\frac{3}{2}}} = \frac{4}{(q^2 - m_+^2)(q^2 - m_-^2)} \frac{d^2\sqrt{P(\zeta)}}{d\zeta^2}$$

In the convolution of two advanced or two retarded distributions the pole integration description has to be changed $q^2 \mp io \to (q \mp io)^2$ everywhere

$$\begin{aligned} \pm \int \frac{d^{1+s}q}{2i\pi} \frac{1}{(q \mp io)^2 - m^2} e^{iqx} &= \int d^{1+s}q \epsilon(q_0) \frac{1 \pm \epsilon(x_0)}{2} \delta(q^2 - m^2) e^{iqx} \\ &= \vartheta(\pm x_0) \int d^{1+s}q \epsilon(q_0) \delta(q^2 - m^2) e^{iqx} \end{aligned}$$

which antisymmetrizes the resulting step function above for the threshold

$$\begin{aligned} &\pm \frac{1}{2i\pi} \frac{1}{(q \mp io)^2 - m_1^2} * \pm \frac{1}{2i\pi} \frac{1}{(q \mp io)^2 - m_2^2} \\ &= \pm \frac{1}{2i\pi} \int_0^1 \delta\zeta \frac{1}{(q \mp io)^2 \zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta)} \\ &= \frac{1}{2\sqrt{|\Delta(q^2)|}} \left[ \epsilon(q_0) \vartheta(q^2 - m_+^2) \mp \frac{1}{i\pi} \{\cdots\} \right] \end{aligned}$$

### 7.3. Residual Product of Representation Functions

The convolutions of causal and Feynman energy–momentum distributions can be summarized with the notation $\overset{R}{*}$ for the different integration contours

$$\mathbf{SO}_0(1,1)\vec{\times}\mathbb{R}^2 : (\overset{R}{*}, q^2) = \begin{cases} (\pm \frac{*}{i\pi}, q^2 \mp io), & \text{Feynman} \\ (\pm \frac{*}{2i\pi}, (q \mp io)^2), & \text{causal} \end{cases}$$

with the results

$$\mathbb{R} : \begin{cases} \frac{\Gamma(1+n_1)}{\left(q^2 - m_1^2\right)^{1+n_1}} \overset{R}{*} \frac{\Gamma(1+n_2)}{\left(q^2 - m_2^2\right)^{1+n_2}} = \int_0^1 d\zeta \frac{\zeta^{n_1}(1-\zeta)^{n_2} \Gamma(1+n_1+n_2)}{\left[q^2\zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta)\right]^{1+n_1+n_2}} \\ \frac{q\Gamma(1+n_1)}{\left(q^2 - m_1^2\right)^{1+n_1}} \overset{R}{*} \frac{\Gamma(1+n_2)}{\left(q^2 - m_2^2\right)^{1+n_2}} = \int_0^1 d\zeta \frac{q\zeta^{n_1}(1-\zeta)^{1+n_2} \Gamma(1+n_1+n_2)}{\left[q^2\zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta)\right]^{1+n_1+n_2}} \\ \frac{q\Gamma(1+n_1)}{\left(q^2 - m_1^2\right)^{1+n_1}} \overset{R}{*} \frac{q\Gamma(1+n_2)}{\left(q^2 - m_2^2\right)^{1+n_2}} = \int_0^1 d\zeta \frac{-\left(\frac{1}{2}\mathbf{1}_2 + q \otimes q \frac{\partial}{\partial q^2}\right)\zeta^{n_1}(1-\zeta)^{n_2} \Gamma(n_1+n_2)}{\left[q^2\zeta(1-\zeta) - m_1^2\zeta - m_2^2(1-\zeta)\right]^{n_1+n_2}} \end{cases}$$

The convolution product contains the normalization factor for the relative energy–momentum residue integral $\frac{1}{2i\pi} \oint$. Therewith, it defines the *residual product* leading from complex representation functions to functions for product representations.

The corresponding residual product for time representations reads

$$\mathbb{R}: \begin{cases} (\overset{R}{*}, q) = (\pm\frac{*}{2i\pi}, q \mp io) \\ \frac{\Gamma(1+n_1)}{(q-m_1)^{1+n_1}} \overset{R}{*} \frac{\Gamma(1+n_2)}{(q-m_2)^{1+n_2}} = \frac{\Gamma(1+n_1+n_2)}{[q-(m_1+m_2)]^{1+n_1+n_2}} \end{cases}$$

The meromorphic functions, i.e., only pole singularities, on the closed complex plane is the field of rational functions. The time representation functions $\mathcal{P}(\overline{\mathbb{C}})$ (pole functions) have negative degree

$$\overline{\mathbb{C}} \ni q \mapsto \frac{P^n(q)}{P^m(q)} = \frac{a_0 + a_1 q + \cdots + a_n q^n}{b_0 + b_1 q + \cdots + b_m q^m} \in \overline{\mathbb{C}}, \quad a_j, b_j \in \mathbb{C},$$
$$b_m \neq 0, \quad n - m \leq -1$$

They have a residual product with unit $\frac{1}{q}$ adding up the invariant singularities.

The $q^2$-singularities for product representations disappear for the residual product of the spacetime representation pole functions. A massless representation function $\frac{q}{q^2}$ has compact invariants only, i.e., a hyperbolic singularity surface. Its residual product

$$\text{time}\,\mathbb{R}: \frac{1}{q} \overset{R}{*} \frac{1}{q-m} = \frac{1}{q-m}$$
$$\text{spacetime}\,\mathbb{R}: \frac{q}{q^2} \overset{R}{*} \frac{q}{(q^2-m^2)^2} = -\left(\frac{1}{2}\mathbf{1}_2 + q \otimes q \frac{\partial}{\partial q^2}\right) \int_0^1 d\zeta \, \frac{1}{q^2\zeta - m^2}$$
$$\int_0^1 d\zeta \, \frac{1}{q^2\zeta - m^2} = \frac{\log \frac{m^2-q^2}{m^2}}{q^2}$$

gives logarithms as integrated representation functions

$$\log\left(1 - \frac{q^2 \pm io}{m^2}\right) = \vartheta(q^2 - m^2)\left[\pm i\pi + \log\left(\frac{q^2}{m^2} - 1\right)\right] + \vartheta(-q^2 + m^2)\log\left(1 - \frac{q^2}{m^2}\right)$$

The logarithm of a quotient is typical for a finite integration (Behnke and Sommer, 1962), e.g., for a function holomorphic on the integration curve

$$\int_a^b dz\; f(z) = \sum \mathrm{Res}\left[f(z)\log\frac{z-b}{z-a}\right], \quad \int_0^\infty dz f(z) = -\sum \mathrm{Res}[f(z)\log z]$$

with the *sum of all residues* in the closed complex plane, cut along the integration curve. For 2-dimensional spacetime $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$ the formulation with the sum of the residues looks as follows

$$-\int_0^1 d\zeta \frac{1}{q^2\zeta - m^2} = \frac{M^2\left(\frac{m^2}{q^2}\right)}{q^2}$$

$$M^2\left(\frac{m^2}{q^2}\right) = \int_0^1 \frac{d\zeta}{\zeta - \frac{m^2}{q^2}} = -\sum \text{Res}\left[\frac{1}{\zeta - \frac{m^2}{q^2}} \log \frac{\zeta - 1}{\zeta}\right] = -\log\left(1 - \frac{q^2}{m^2}\right)$$

## 8. RESIDUAL REPRESENTATIONS OF FOUR-DIMENSIONAL SPACETIME

Four-dimensional Minkowski spacetime and its Lorentz group has—with 3D position translations $\mathbb{R}^3$—additional rotation degrees of freedom from the 2-sphere $\Omega^2$. Spacetime is used in the Cartan representation with hermitian complex $(2 \times 2)$-matrices where the trace is the time projection

$$\mathbb{R} \oplus [\mathbb{R}_\vee \times \mathbb{I}(2)] \cong \mathbb{R}^2 \hookrightarrow \mathbb{R}^4 \cong \mathbb{R} \oplus [\mathbb{R}_\vee \times \Omega^2]$$

$$\begin{pmatrix} x_0 + x_3 & 0 \\ 0 & x_0 - x_3 \end{pmatrix} \hookrightarrow \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

$$= u\left(\frac{\vec{x}}{r}\right)\begin{pmatrix} x_0 + r & 0 \\ 0 & x_0 - r \end{pmatrix} u^*\left(\frac{\vec{x}}{r}\right)$$

$$\mathbf{SO}_0(1,1) \hookrightarrow \mathbf{SO}_0(1,3) \cong \mathbf{SO}_0(1,1) \times \Omega^2 \times \mathbf{SO}(2) \times \Omega^2$$

It requires rotation representations that will lead, in comparison to 2-dimensional spacetime, to a change in the pole orders for residual representations.

### 8.1. Feynman Distributions

In the Feynman and causal energy–momentum distributions

$$\text{Feynman:} \mp \frac{1}{i\pi^2} \frac{\Gamma(2+n)}{(q^2 \mp io - m^2)^{2+n}} = -\frac{1}{\pi}\delta^{(1+n)}(m^2 - q^2) \mp \frac{1}{i\pi^2} \frac{\Gamma(2+n)}{\left(q_{\text{P}}^2 - m^2\right)^{2+n}}$$

$$\text{for} \quad n = -1, 0, 1, \ldots$$

$$\text{Causal:} \mp \frac{1}{2i\pi^2} \frac{\Gamma(2+n)}{((q \mp io)^2 - m^2)^{2+n}} = -\frac{1}{2\pi}\epsilon(q_0)\delta^{(1+n)}(m^2 - q^2)$$

$$\mp \frac{1}{2i\pi^2} \frac{\Gamma(2+n)}{\left(q_{\text{P}}^2 - m^2\right)^{2+n}}$$

$$\text{with} \quad (q \mp io)^2 = (q_0 \mp io)^2 - \vec{q}^2$$

there is an additional residual normalization factor $-\frac{1}{\pi}$ for the 2-sphere.

The Fourier transformations $d^4q = dq_0 d\Omega^2 \vec{q}^2 d|\vec{q}|$ in 4D spacetime are obtainable from the 2D case by an invariant derivation (2-sphere spread)

$$\begin{aligned}\int \frac{d^4q}{4\pi} \begin{pmatrix} 1 \\ \epsilon(q_0)\vartheta(q^2) \end{pmatrix} \tilde{\mu}(q^2)e^{iqx} &= -\frac{\partial}{\partial r^2} \int dq_0 dq_3 \begin{pmatrix} 1 \\ \epsilon(q_0)\vartheta\left(q_0^2 - q_3^2\right) \end{pmatrix} \\ &\quad \times \tilde{\mu}\left(q_0^2 - q_3^2\right) e^{iq_0x_0 - iq_3r} \\ &= \frac{\partial}{\partial x^2} \int d^2q \begin{pmatrix} 1 \\ \epsilon(q_0)\vartheta(q^2) \end{pmatrix} \tilde{\mu}(q^2)e^{iqx}|_{x=(x_0,r)}\end{aligned}$$

One obtains as Fourier transformation of the Dirac distribution

$$\int \frac{d^4q}{\pi} \delta(q^2 - m^2)e^{iqx} = -\frac{\partial}{\partial \frac{x^2}{4}} \mathcal{N}_0 \left( \sqrt{\frac{m^2x^2}{4}} \right)$$

and the causally supported Fourier transforms

$$\begin{aligned}\int \frac{d^4q}{i\pi^2} \frac{\Gamma(2+n)}{\left(q_{\mathrm{P}}^2 - m^2\right)^{2+n}} e^{iqx} &= \epsilon(x_0) \int \frac{d^4q}{2\pi} \epsilon(q_0)\delta^{(1+n)}(m^2 - q^2)e^{iqx} \\ &= i\pi \left( \frac{d}{dm^2} \right)^{1+n} \frac{\partial}{\partial \frac{x^2}{4}} \vartheta(x^2)\varepsilon_0 \left( \frac{m^2x^2}{4} \right) \\ &= \begin{cases} i\pi \left[ \delta\left(\frac{x^2}{4}\right) - \vartheta(x^2)m^2\varepsilon_1\left(\frac{m^2x^2}{4}\right) \right], & n = -1 \\ -i\pi\vartheta(x^2)\left(-\frac{x^2}{4}\right)^n \varepsilon_n\left(\frac{m^2x^2}{4}\right), & n = 0, 1, \ldots \end{cases}\end{aligned}$$

The Kepler (Yukawa) factor $\frac{1}{r}$-singularity is embedded into the lightcone Dirac distribution $\frac{\partial}{\partial x^2}\vartheta(x^2) = \delta(x^2)$ for the simple pole $n = -1$.

Feynman propagators of scalar particle fields come with simple poles.

## 8.2. Time and Position Frames

By partial Fourier transformation with respect to energy and momentum one obtains the embedded time and position representations

$$\begin{aligned}\pi g(m^2, x) &= \int \frac{d^4q}{\pi} e^{iqx} \tilde{g}(m^2, q) = \int \frac{d^3q}{\pi} e^{-i\vec{q}\vec{x}} g(q_0, x_0) \\ &= \int dq_0\, e^{iq_0x_0} \left[ \vartheta\left(q_0^2 - m^2\right) g^c(|\vec{q}|, \vec{x}) + \vartheta\left(m^2 - q_0^2\right) g^{nc}(i|\vec{q}, \vec{x}) \right]\end{aligned}$$

Time and Position Representations for $\mathbb{R}^4$-Spacetime

| | Time | Position (compact) | Position (noncompact) |
|---|---|---|---|
| $\tilde{g}(m^2, q)$ | $g(q_0, x_0)$ | $g^c(\lvert\vec{q}\rvert, \vec{x})$ | $g^{nc}(i\lvert\vec{q}\rvert, \vec{x})$ |
| | $q_0 = \sqrt{m^2 + \vec{q}^2}$ | $\lvert\vec{q}\rvert = \sqrt{q_0^2 - m^2}$ | $i\lvert\vec{q}\rvert = \lvert Q\rvert = \sqrt{m^2 - q_0^2}$ |
| *Lorentz scalars* | | | |
| $\delta(m^2 - q^2)$ | $\frac{\cos q_0 x_0}{q_0}$ | $2\frac{\sin\lvert\vec{q}\rvert r}{r}$ | 0 |
| $\epsilon(q_0)\delta(m^2 - q^2)$ | $i\frac{\sin q_0 x_0}{q_0}$ | $2\epsilon(q_0)\frac{\sin\lvert\vec{q}\rvert r}{r}$ | 0 |
| $\epsilon(q_0)\delta'(m^2 - q^2)$ | $\frac{x_0^2}{2q_0} i j_1(q_0 x_0)$ | $\epsilon(q_0)\frac{\cos\lvert\vec{q}\rvert r}{\lvert\vec{q}\rvert}$ | 0 |
| $\frac{1}{i\pi}\frac{1}{q_{\rm P}^2 - m^2}$ | $\epsilon(x_0) i \frac{\sin q_0 x_0}{q_0}$ | $\bullet 2i\frac{\cos\lvert\vec{q}\rvert r}{r}$ | $\bullet 2i\frac{e-\lvert Q\rvert r}{r}$ |
| $\frac{1}{i\pi}\frac{1}{\left(q_{\rm P}^2 - m^2\right)^2}$ | $\epsilon(x_0)\frac{x_0^2}{2q_0} i j_1(q_0 x_0)$ | $-i\frac{\sin\lvert\vec{q}\rvert r}{\lvert\vec{q}\rvert}$ | $-i\frac{e-\lvert Q\rvert r}{\lvert Q\rvert}$ |
| *Lorentz vectors* | | | |
| $q\delta(m^2 - q^2)$ | $\begin{pmatrix} i\sin q_0 x_0 \\ \frac{\vec{q}}{q_0}\cos q_0 x_0 \end{pmatrix}$ | $2\vec{q}^2\begin{pmatrix} \frac{q_0}{\lvert\vec{q}\rvert} j_0(\lvert\vec{q}\rvert r) \\ \frac{\vec{x}}{r} i j_1(\lvert\vec{q}\rvert r) \end{pmatrix}$ | 0 |
| $q\epsilon(q_0)\delta(m^2 - q^2)$ | $\begin{pmatrix} \cos q_0 x_0 \\ \frac{\vec{q}}{q_0} i\sin q_0 x_0 \end{pmatrix}$ | $\epsilon(q_0) 2\vec{q}^2\begin{pmatrix} \frac{q_0}{\lvert\vec{q}\rvert} j_0(\lvert\vec{q}\rvert r) \\ \frac{\vec{x}}{r} i j_1(\lvert\vec{q}\rvert r) \end{pmatrix}$ | 0 |
| $q\epsilon(q_0)\delta'(m^2 - q^2)$ | $-\frac{x_0^2}{2}\begin{pmatrix} j_0(q_0 x_0) \\ \frac{\vec{q}}{q_0} i j_1(q_0 x_0) \end{pmatrix}$ | $\epsilon(q_0)\begin{pmatrix} \frac{q_0}{\lvert\vec{q}\rvert}\cos\lvert\vec{q}\rvert r \\ \frac{\vec{x}}{r} i\sin\lvert\vec{q}\rvert r \end{pmatrix}$ | 0 |
| $\frac{1}{i\pi}\frac{q}{q_{\rm P}^2 - m^2}$ | $\epsilon(x_0)\begin{pmatrix} \cos q_0 x_0 \\ \frac{\vec{q}}{q_0} i\sin q_0 x_0 \end{pmatrix}$ | $\bullet 2\vec{q}^2\begin{pmatrix} \frac{q_0}{\lvert\vec{q}\rvert} i n_0(\lvert\vec{q}\rvert r) \\ -\frac{\vec{x}}{r} n_1(\lvert\vec{q}\rvert r) \end{pmatrix}$ | $\bullet 2\lvert Q\rvert^2\begin{pmatrix} -i\frac{q_0}{\lvert Q\rvert} k_0(\lvert Q\rvert r) \\ \frac{\vec{x}}{r} k_1(\lvert Q\rvert r) \end{pmatrix}$ |
| $\frac{1}{i\pi}\frac{q}{\left(q_{\rm P}^2 - m^2\right)^2}$ | $-\epsilon(x_0)\frac{x_0^2}{2}\begin{pmatrix} j_0(q_0 x_0) \\ \frac{\vec{q}}{q_0} i j_1(q_0 x_0) \end{pmatrix}$ | $\circ - \begin{pmatrix} \frac{q_0}{\lvert\vec{q}\rvert} i\sin\lvert\vec{q}\rvert r \\ \frac{\vec{x}}{r}\cos\lvert\vec{q}\rvert r \end{pmatrix}$ | $\circ - \begin{pmatrix} i\frac{q_0}{\lvert Q\rvert} \\ \frac{\vec{x}}{r} \end{pmatrix} e^{-\lvert Q\rvert r}$ |

with—for higher order poles

$$\frac{\partial}{\partial|m|} = 2|m|\frac{\partial}{\partial m^2} \cong \frac{|m|}{q_0}\frac{\partial}{\partial q_0} \cong -\frac{|m|}{|\vec{q}|}\frac{\partial}{\partial|\vec{q}|} \cong \frac{|m|}{|Q|}\frac{\partial}{\partial|Q|}$$

There arise the scalar and vector Bessel functions $j_L = 0, 1$ (spherical waves for free particles), Neumann functions $n_L$, and Macdonald functions $k_L$ (for Yukawa interactions and forces). $r = 0$-singular and $r = 0$-ambiguous elements $\frac{\vec{x}}{r}$ which are no position representations come with simple and double poles and are marked with $\bullet$ and $\circ$ resp.

With the embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ the time representations $\mathbb{R} \hookrightarrow \mathbb{R}$ remain simple poles, the position representation $\mathbb{R} \hookrightarrow \mathbb{R}^3$ come as scalar dipoles and vectorial tripoles as seen in the projections for trivial momenta $\vec{q} = 0$ (time frame) and trivial energy $q_0 = 0$ (position frame with "imaginary" momenta), respectively.

Time and Position Projection for $\mathbb{R}^4 \cong \mathbb{R} \oplus \mathbb{R}^3$

| | Time frame ($x_0 = t$) | Position frame |
|---|---|---|
| $\tilde{g}(m^2, q)$ | $g(\|m\|, t) = \int \frac{d^3x}{8\pi} g(m^2, x)$ <br> $(q_0, \vec{q}) = (\|m\|, 0)$ | $g^{nc}(i\|m\|, \vec{x}) = \int \frac{dx_0}{2} g(m^2, x)$ <br> $(q_0, \|\vec{q}\|) = (0, i\|m\|)$ |
| *Lorentz scalars* | | |
| $\delta(m^2 - q^2)$ | $\frac{\cos mt}{\|m\|}$ | $0$ |
| $\epsilon(q_0)\delta(m^2 - q^2)$ | $i\,\frac{\sin mt}{m}$ | $0$ |
| $\frac{1}{i\pi}\frac{1}{q_P^2 - m^2}$ | $\epsilon(t) i\,\frac{\sin mt}{m}$ | $\bullet 2i\,\frac{e^{-\|m\|r}}{r}$ |
| $\frac{1}{i\pi}\frac{1}{\left(q_P^2 - m^2\right)^2}$ | $\epsilon(t) i\,\frac{\sin \|m\|t - \|m\|t \cos mt}{2\|m\|^3}$ | $-i\,\frac{e^{-\|m\|r}}{r}$ |
| *Lorentz vectors* | | |
| $q\delta(m^2 - q^2)$ | $\begin{pmatrix} i \sin \|m\|t \\ 0 \end{pmatrix}$ | $0$ |
| $q\epsilon(q_0)\delta(m^2 - q^2)$ | $\begin{pmatrix} \cos mt \\ 0 \end{pmatrix}$ | $0$ |
| $\frac{1}{i\pi}\frac{q}{q_P^2 - m^2}$ | $\epsilon(t)\begin{pmatrix} \cos mt \\ 0 \end{pmatrix}$ | $\bullet \begin{pmatrix} 0 \\ 2\frac{\vec{x}}{r}\frac{1+\|m\|r}{r^2} e^{-\|m\|r} \end{pmatrix}$ |
| $\frac{1}{i\pi}\frac{q}{\left(q_P^2 - m^2\right)^2}$ | $\epsilon(t)\begin{pmatrix} -\frac{t \sin \|m\|t}{2\|m\|} \\ 0 \end{pmatrix}$ | $\circ \begin{pmatrix} 0 \\ -\frac{\vec{x}}{r} \end{pmatrix} e^{-\|m\|r}$ |
| $\frac{1}{i\pi}\frac{q}{\left(q_P^2 - m^2\right)^3}$ | $\epsilon(t)\begin{pmatrix} \frac{t \sin \|mt\| - \|mt\| \cos mt}{4\|m\|^3} \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ \vec{x} \end{pmatrix} \frac{e^{-\|m\|r}}{2\|m\|}$ |

The time representations from the Dirac and principal value distributions have nildimensions $N = 0, 1, 2$ for poles, dipoles, tripoles. The $r = 0$ regular nonambiguous position representation matrix elements are the knotless Kepler bound state wave functions above, embedded into the principal value energy–momentum distributions for spacetime representations with timelike support $x^2 > 0$

$$|1, \vec{0}\rangle \sim e^{-|m|r} \hookrightarrow \int \frac{d^4q}{\pi^2} \frac{|m|}{\left(q_P^2 - m^2\right)^2} e^{iqx}$$

$$|2, \vec{1}\rangle \sim 2|m|\vec{x}e^{-|m|r} \hookrightarrow \int \frac{d^4q}{i\pi^2} \frac{4m^2 q}{\left(q_P^2 - m^2\right)^3} e^{iqx}$$

The complex representation functions for 4-dimensional spacetime, e.g.

$$\overline{\mathbb{C}}^2 \times \Omega^2 \ni q \mapsto \frac{q}{q^2 - m^2} \in \overline{\mathbb{C}}^2 \times \Omega^2$$

give as energy and momentum projected residues for the Lorentz scalar functions

$$\operatorname*{Res}_{\pm i|m|} \frac{1}{q^2 - m^2} = \oint_{\pm|m|} \frac{d^4q}{2i\pi^2} \pi\delta(\vec{q}) \frac{1}{q^2 - m^2} = \oint_{\pm|m|} \frac{dq}{2i\pi} \frac{1}{q^2 - m^2} = \pm\frac{1}{2|m|}$$

$$\operatorname*{Res}_{\pm i|m|} \frac{1}{q^2 - m^2} = \oint_{\pm i|m|} \frac{d^4q}{2i\pi^2} \delta(q_0) \frac{1}{q^2 - m^2} = -\oint_{\pm i|m|} \frac{d^3q}{2i\pi^2} \frac{1}{q^2 + m^2} = \mp \frac{i|m|}{2}$$

and for the Lorentz vector $q = q_0 \mathbf{1}_2 + \vec{q}$ a trace residue tr $q = 2q_0$ for the energy projection

$$\operatorname{tr} \operatorname*{Res}_{\pm i|m|} \frac{q}{q^2 - m^2} = \oint_{\pm|m|} \frac{dq}{2i\pi} \frac{2q^3}{q^2 - m^2} = m^2$$

## 8.3. Residual Products (Feynman Integrals)

Pointwise products of Feynman propagators convolute energy–momentum distributions which, in general however, are not convolutable. For particle propagators, there arise undefined local products ("divergencies") of generalized functions from the imaginary principal value for the causally supported part $\frac{1}{i\pi} \frac{1}{q_{\mathrm{P}}^2 - m^2} \sim i\pi\delta(x^2) + \cdots$

$$\begin{aligned} &\left[ -\frac{1}{x_{\mathrm{P}}^2} + i\pi\delta(x^2) + \cdots \right] \bullet \left[ -\frac{1}{x_{\mathrm{P}}^2} + i\pi\delta(x^2) + \cdots \right] \\ &\sim \left[ \delta\left(q^2 - m_1^2\right) + \frac{1}{i\pi} \frac{1}{q_{\mathrm{P}}^2 - m_1^2} \right] * \left[ \delta\left(q^2 - m_2^2\right) + \frac{1}{i\pi} \frac{1}{q_{\mathrm{P}}^2 - m_1^2} \right] \end{aligned}$$

The convolution of two Feynman distributions

$$\begin{aligned} &\mp \frac{1}{i\pi^2} \frac{\binom{1}{q}\Gamma(2+n_1)}{(q^2 \mp io - m_1^2)^{2+n_1}} * \mp \frac{1}{i\pi^2} \frac{\binom{1}{q}\Gamma(2+n_2)}{(q^2 \mp io - m_1^2)^{2+n_2}} \\ &= \frac{1}{i\pi^2} \int_0^1 d\zeta_{1,2} \delta(\zeta_1 + \zeta_2 - 1) \\ &\quad \int \frac{d^4p}{i\pi^2} \frac{\begin{pmatrix} 1 & q\zeta_1 \\ q\zeta_2 & -p \otimes p + q \otimes q\zeta_1\zeta_2 \end{pmatrix} \zeta_1^{1+n_1} \zeta_2^{1+n_2} \Gamma(4 + n_1 + n_2)}{\left[p^2 \mp io + q^2\zeta_1\zeta_2 - m_1^2\zeta_1 - m_2^2\zeta_2\right]^{4+n_1+n_2}} \end{aligned}$$

involves the tensor $p \otimes p - q \otimes q\zeta_1\zeta_2 \Rightarrow \frac{p^2}{4}\mathbf{1}_4 - q \otimes q\zeta_1\zeta_2$ for the vector–vector convolution. Taking the $q$-dependent residue of the relative energy–momenta

$$\mp \int \frac{d^4p}{i\pi^2} \frac{\Gamma(3+n)}{(p^2 \mp io + a)^{3+n}} = \mp \frac{1}{2} \int \frac{d^4p}{i\pi^2} \frac{p^2\Gamma(4+n)}{(p^2 \mp io + a)^{4+n}} = \frac{\Gamma(1+n)}{(\mp io + a)^{1+n}}$$

and with the notation for the different contours

$$\mathbf{SO}_0(1,3)\vec{\times}\mathbb{R}^4 : (\overset{R}{*}, q^2) = \begin{cases} \left(\mp\frac{*}{i\pi^2}, q^2 \mp io\right), & \text{Feynman} \\ \left(\mp\frac{*}{2i\pi^2}, (q^2 \mp io)^2\right), & \text{casual} \end{cases}$$

the residual scalar–scalar, vector–scalar, and vector–vector product reads

$$\mathbb{R}^4 : \begin{cases} \frac{\Gamma(2+n_1)}{\left(q^2-m_1^2\right)^{2+n_1}} \overset{R}{*} \frac{\Gamma(2+n_1)}{\left(q^2-m_2^2\right)^{2+n_1}} = \int_0^1 d\zeta \, \frac{\zeta^{1+n_1}(1-\zeta)^{1+n_2}\Gamma(2+n_1+n_2)}{\left[q^2\zeta(1-\zeta)-m_1^2\zeta-m_2^2(1-\zeta)\right]^{2+n_1+n_2}} \\ \frac{q\Gamma(2+n_1)}{\left(q^2-m_1^2\right)^{2+n_1}} \overset{R}{*} \frac{\Gamma(2+n_2)}{\left(q^2-m_2^2\right)^{2+n_1}} = \int_0^1 d\zeta \, \frac{q\zeta^{1+n_1}(1-\zeta)^{2+n_2}\Gamma(2+n_1+n_2)}{\left[q^2\zeta(1-\zeta)-m_1^2\zeta-m_2^2(1-\zeta)\right]^{2+n_1+n_2}} \\ \frac{q\Gamma(2+n_1)}{\left(q^2-m_1^2\right)^{2+n_1}} \overset{R}{*} \frac{q\Gamma(2+n_2)}{\left(q^2-m_2^2\right)^{2+n_1}} = \int_0^1 d\zeta \, \frac{-\left(\frac{1}{2}\mathbf{1}_4+q\otimes q\frac{\partial}{\partial q^2}\right)\zeta^{1+n_1}(1-\zeta)^{1+n_2}\Gamma(1+n_1+n_2)}{\left[q^2\zeta(1-\zeta)-m_1^2\zeta-m_2^2(1-\zeta)\right]^{1+n_1+n_2}} \end{cases}$$

The $q^2$-poles in the residual products for the energy and momentum rational complex functions disappear in the residual product of the energy–momentum pole functions

$$\frac{q}{q^2} \overset{R}{*} \frac{2q}{(q^2-m^2)^3} = -\left(\frac{1}{2}\mathbf{1}_4 + q\otimes q\frac{\partial}{\partial q^2}\right)\frac{1}{q^2}\int_0^1 d\zeta \, \frac{1-\zeta}{\zeta-\frac{m^2}{q^2}}$$

$$\frac{q}{(q^2)^2} \overset{R}{*} \frac{2q}{(q^2-m^2)^3} = -\left(\frac{1}{2}\mathbf{1}_4 + q\otimes q\frac{\partial}{\partial q^2}\right)\frac{1}{(q^2)^2}\int_0^1 d\zeta \, \frac{\zeta}{\left(\zeta-\frac{m^2}{q^2}\right)^2}$$

with the residue sum in the closed complex plane (there is a nontrivial residue at the holomorphic point $\zeta = \infty$), e.g.

$$M^2\left(\frac{m^2}{q^2}\right) = -\int_0^1 d\zeta \, \frac{1-\zeta}{\zeta-\frac{m^2}{q^2}} = -\sum \mathrm{Res}\frac{1-\zeta}{\zeta-\frac{m^2}{q^2}}\mathrm{Log}\frac{\zeta-1}{\zeta}$$
$$= 1-\left(1-\frac{m^2}{q^2}\right)\log\left(1-\frac{q^2}{m^2}\right)$$

## 9. LORENTZ COMPATIBLE SPIN EMBEDDING

The embedding of position representations into Minkowski spacetime has to embed the harmonic momentum polynomials $(\vec{q})^{2J} = |\vec{q}|^{2J}Y^{2J}(\varphi,\theta)$ and has to interpret this embedding with respect to time representations involved.

The connection between spin $\mathbf{SO}(3)$ and its covering $\mathbf{SU}(2)$ to the Lorentz group $\mathbf{SO}_0(1,3)$ with its covering $\mathbf{SL}(\mathbb{C}^2)$ is given by transmutators as representatives of the symmetric space $\mathbf{SL}\,(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$, i.e., of the orientation manifold of the spin group. All those transmutators (boost representations) are products of the two $(2 \times 2)$ transmutators from Pauli spinors $V \cong \mathbb{C}^2$ to left and right handed Weyl spinor $V_L \cong \mathbb{C}^2 \cong V_R$

$$s(\underline{q}) : V \mapsto V_L, \quad \hat{s}(\underline{q}) : V \mapsto V_R, \quad \hat{s} = S^{-1*}$$

parametrizable with normalized positive energy–momenta

$$q^2 > 0, \quad \frac{q}{\sqrt{q^2}} = \underline{q} = \underline{q}_0 \mathbf{1}_2 + \underline{\vec{q}} = \begin{pmatrix} \underline{q}_0 + \underline{q}_3 & \underline{q}_1 - i\underline{q}_2 \\ \underline{q}_1 + i\underline{q}_2 & \underline{q}_0 - i\underline{q}_3 \end{pmatrix}$$
$$\underline{\check{q}} = \underline{q}_0 \mathbf{1}_2 + \underline{\vec{q}}, \quad \underline{q}^2 = 1 = \underline{\check{q}}^2$$

Both Weyl transmutators embed the unit $\mathbf{1}_2$ for the Pauli spinor space and the spherical harmonics $Y^1(\varphi, \theta) = \frac{\vec{q}}{|\vec{q}|}$ into the normalized energy–momenta

$$s(\underline{q})\mathbf{1}_2 s^*(\underline{q}) = \underline{q}, \quad \hat{s}(\underline{q})\mathbf{1}_2 \hat{s}^*(\underline{q}) = \underline{\check{q}}, \quad s(\underline{q}), \hat{s}(\underline{q}) = s(\underline{\check{q}}) \in \mathbf{SL}(\mathbb{C}^2)$$
$$\Rightarrow s(\underline{q}) = u\left(\frac{\vec{q}}{|\vec{q}|}\right) \circ e^{\frac{\beta}{2}\sigma 3} \circ u^*\left(\frac{\vec{q}}{|\vec{q}|}\right), \quad \tanh\beta = \frac{q0}{|\vec{q}|}, \quad u\left(\frac{\vec{q}}{|\vec{q}|}\right) \in \mathbf{SU}(2)$$
$$= \frac{1}{\sqrt{2(1+\underline{q}_0)}} \begin{pmatrix} 1 + \underline{q}_0 + \underline{q}_3 & \underline{q}_1 - i\underline{q}_2 \\ \underline{q}_1 + i\underline{q}_2 & 1 + \underline{q}_0 - \underline{q}_3 \end{pmatrix}$$

Now the general case: an $\mathbf{SU}(2)$-representation $[2J]$ with spin $J = 0, \frac{1}{2}, \ldots$ is embedded into finite dimensional irreducible representations $[2L|2R]$ with left and right "spin" $L, R$ of the Lorentz group $\mathbf{SL}(\mathbb{C}^2)$ for

$$[2J] \hookrightarrow [2L|2R] \quad \text{for} \quad \begin{cases} L + R \geq J \\ L + R + J \text{integer} \end{cases}$$

with the $\mathbf{SU}(2)$-decomposition

$$[2L|2R] \cong \bigoplus_{J=|L-R|}^{L+R} [2J], \quad \mathbb{C}^{(1+2L)(1+2R)} \cong \bigoplus_{J=|L-R|}^{L+R} \mathbb{C}^{1+2J}$$

The Lorentz group acts upon the totally symmetrized products $\bigvee^{2L} V_L \otimes \bigvee^{2R} V_R \cong \mathbb{C}^{(1+2L)(1+2R)}$ of Weyl spaces. The transmutators

$$s^{[2L|2R]}(\underline{q}) = \bigvee^{2L} s(\underline{q}) \otimes \bigvee^{2R} \hat{s}(\underline{q}) : \bigvee^{2L} V \otimes \bigvee^{2R} V \to \bigvee^{2L} V_L \otimes \bigvee^{2R} V_R$$

allow the Lorentz compatible embedding of spin properties.

For example, the Minkowski representation of the boosts

$$s^{[1|1]}(\underline{q}) \;:\; V \otimes V \to V_L \otimes V_R$$

$$s^{[1|1]}(\underline{q}) = s(\underline{q}) \otimes \hat{s}(\underline{q}) = \Lambda(\underline{q}) = \begin{pmatrix} \underline{q}_0 & \underline{\vec{q}} \\ \underline{\vec{q}} & \mathbf{1}_3 + \frac{\underline{\vec{q}} \otimes \underline{\vec{q}}}{1+\underline{q}_0} \end{pmatrix} \in \mathbf{SO}_0(1, 3)$$

gives the Lorentz compatible embeddings with the projectors for spin 0 and 1

$$V_L \otimes V_R \cong \mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C}^3, \begin{cases} \Lambda(\underline{q}) \left(\begin{array}{c|c} 1 & 0 \\ 0 & 0 \end{array}\right) \Lambda^{-1}(\underline{q}) &= \underline{q} \otimes \check{\underline{q}} \cong \frac{q^j qk}{q^2} \\ \Lambda(\underline{q}) \left(\begin{array}{c|c} 0 & 0 \\ 0 & \mathbf{1}_3 \end{array}\right) \Lambda^{-1}(\underline{q}) &= \mathbf{1}_4 - \underline{q} \otimes \check{\underline{q}} \cong \delta_k^j - \frac{q^j qk}{q^2} \end{cases}$$

This example is characteristic: the totally symmetric spherical harmonics are embedded for integer spin in symmetric $\mathbf{SO}_0(1, 3)$-representations

$$j = 0, 1, \ldots : [2J] \hookrightarrow [2L|2L] \quad \text{with} \quad 2L \geq J$$

$$\left(\frac{\vec{q}}{|\vec{q}|}\right)^{2J} \hookrightarrow (\underline{q})^{4L}_{2J} = \bigvee^{2L} \underline{q} \overset{2J}{\otimes} \bigvee^{2L} \check{\underline{q}}$$

with the decomposition of the unit matrix into projectors

$$J = 0, 1, \ldots : \mathbf{1}_{(1+2L)^2} = \bigoplus_{J=0}^{2L} (\underline{q})^{4L}_{2J}, \quad (\underline{q})^{4L}_{2J} = s^{[2L|2L]}(\underline{q}) \mathbf{1}_{1+2J} s^{-1[2L|2L]}(\underline{q})$$

$$(\underline{q})^{4L}_{2J} \circ (\underline{q})^{4L}_{2J'} = \delta_{JJ'} (\underline{q})^{4L}_{2J}$$

In generalization of the two Weyl representations there arise two embedding types for half-integer spin, conjugated to each other. They can be Clebsch–Gordan composed from the two Weyl transmutators

$$J = \frac{1}{2}, \frac{1}{3}, \ldots : [2J] \hookrightarrow \begin{cases} [1 + 2L|2L] \\ [2L|1 + 2L] \end{cases} \quad \text{with} \quad 2L \geq J - \frac{1}{2}$$

$$\left(\frac{\vec{q}}{|\vec{q}|}\right)^{2J} \hookrightarrow \begin{cases} (\underline{q})^{1+4L}_{2J} = \bigvee^{1+2L} \underline{q} \overset{2J}{\otimes} \bigvee^{2L} \check{\underline{q}} \\ (\check{\underline{q}})^{1+4L}_{2J} = \bigvee^{2L} \underline{q} \overset{2J}{\otimes} \bigvee^{1+2L} \check{\underline{q}} \end{cases}$$

An appropriate $\mathbf{D}(1)$-dilatation factor gives transmutators from $\mathbf{U}(2)$ to $\mathbf{GL}(\mathbb{C}^2)$, i.e., representatives of the symmetric space $\mathbf{GL}\mathbb{C}^2/\mathbf{U}(2)$

$$q^2 \geq 0 : s(q) = \sqrt{q^2} s(\underline{q}) = u\left(\frac{\vec{q}}{|\vec{q}|}\right) \circ \sqrt{q^2} e^{\frac{\beta}{2}\sigma_3} \circ u^* \left(\frac{\vec{q}}{|\vec{q}|}\right) \in \mathbf{GL}(\mathbb{C}^2)$$

Therewith the harmonic polynomials are Lorentz compatibly embedded

$$\left(\frac{\vec{q}}{|\vec{q}|}\right)^{2J} \hookrightarrow (\underline{q})^K_{2J}, \quad (\vec{q})^{2J} \hookrightarrow (q)^N_{2J} = (\sqrt{q^2})^{2K}(\underline{q})^N_{2J}$$

with the examples from above for Lorentz scalar, left and right Weyl spinor and Lorentz vector (with the projectors $\mathbf{1}_4 = \mathbf{1}_1 + \mathbf{1}_3$)

$$(\vec{q})^0 \hookrightarrow (q)^0_0 = 1, \qquad (\vec{q})^1 \hookrightarrow \begin{cases} (q)^1_1 = q = s(q)\mathbf{1}_2 s^*(q) \\ (\check{q})^1_1 = \check{q} = \hat{s}(q)\mathbf{1}_2 \hat{s}^*(q) \end{cases}$$

Convolutions of energy–momenta are understood to involve also the tensor products of the spin representations. For example, in the vector–vector convolution above there arises the projectors for spin 0 and 1

$$q * \check{q} \Rightarrow \frac{1}{2}\mathbf{1}_4 + q \otimes \check{q}\frac{\partial}{\partial q^2} = \frac{1}{2}(\underline{q})^2_2 + \left[\frac{1}{2} + q^2\frac{\partial}{\partial q^2}\right](\underline{q})^2_0$$

## 10. RESIDUAL REPRESENTATIONS OF FUTURE CONES

Causal (advanced and retarded) and Feynman multipole energy–momentum distributions lead—via their Fourier transforms with appropriate integration contours—to representation matrix elements of different symmetric spaces—of the causal bicone (future and past cone) and of the tangent spacetime translations, respectively. Feynman distributions with $\delta(q^2 - m^2)$ from a simple pole represent spacetime translations as inhomogeneous subgroup of irreducible unitary Poincaré group representations, acting on free particles. The representations of the future cone as model of nonlinear spacetime (Saller, 1999, 2001b) involves higher order energy–momentum poles. They are no particle propagators. They will be used to determine the masses and normalization of particles for the construction of Feynman propagators.

### 10.1. Spacetime Future Cones

One dimensional time future is embedded into the future cones of 2D and 4D Minkowski spacetime

$$\mathbb{R}_\vee \ni t_\vee = \vartheta(t)t \hookrightarrow \vartheta(x^0)\vartheta(x^2)\begin{pmatrix} x^0 + x^3 & 0 \\ 0 & x^0 - x^3 \end{pmatrix} = x_\vee \in \mathbb{R}^2_\vee$$

$$\hookrightarrow \vartheta(x^0)\vartheta(x^2)\begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x_\vee \in \mathbb{R}^4_\vee$$

with associated orthochronous groups—trivial, abelian, simple

$$\{1\} = \mathbf{SO}(1) \hookrightarrow \mathbf{SO}_0(1,1) \hookrightarrow \mathbf{SO}_0(1,3)$$

Time future is the causal group $\mathbf{D}(1) = \exp \mathbb{R}$

$$\mathbb{R}_\vee \ni t_\vee = e^{\psi^0} \in \mathbf{D}(1)$$
$$\mathbb{R}_\vee \cong \mathbf{D}(1) \cong \mathbf{GL}(\mathbb{C})/\mathbf{U}(1)$$

The 2-dimensional future cone is the direct product of causal group and selfdual Lorentz dilatation group

$$\mathbb{R}^2_\vee \ni x_\vee = \begin{pmatrix} x^0_\vee + x^3 & 0 \\ 0 & x^0_\vee - x^3 \end{pmatrix} = e^{\psi^0 + \sigma 3 \psi^3} \quad \text{with} \quad \begin{cases} x^2_\vee &= e^{2\psi^0} \\ \frac{x^0_\vee + x^3}{x^0_\vee - x^3} &= e^{2\psi^3} \end{cases}$$
$$\mathbb{R}^2_\vee \cong \mathbf{D}(1) \times \mathbf{SO}_0(1,1)$$

The 4-dimensional future cone is a homogeneous space with 2-dimensional future $\mathbb{R}^2_\vee$ as abelian Cartan substructure

$$\mathbb{R}^4_\vee \ni x_\vee = \begin{pmatrix} x^0_\vee + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0_\vee - x^3 \end{pmatrix} = e^{\psi^0 + \vec{\psi}} = u\left(\frac{\vec{x}}{r}\right) \circ e^{\psi^0 + \sigma 3 |\psi|} \circ u\left(\frac{\vec{x}}{r}\right)^*$$
$$\text{with} \quad \begin{cases} x^2_\vee = e^{2\psi^0}, & \frac{x^0_\vee + r}{x^0_\vee - r} = e^{2|\vec{\psi}|} \\ \frac{\vec{\psi}}{|\vec{\psi}|} = \frac{\vec{x}}{r}, & u(\frac{\vec{x}}{r}) \in \mathbf{SU}(2) \end{cases}$$

$$\begin{aligned} \mathbb{R}^4_\vee &\cong \mathbf{D}(1) \times \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \\ &\cong \mathbf{D}(1) \times \mathbf{SO}_0(1,1) \times \Omega^2 \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \end{aligned}$$

The cones as irreducible orbits of $\mathbf{D}(1) \times \mathbf{SO}_0\ (\mathbf{1}, s)$, $s = 0, 1, 3$ are used as strict futures, open without "skin," i.e., without the strict presence $x = 0$ and without lightlike translations for nontrivial position $s = 1, 3$

$$x_\vee \in \mathbb{R}^{1+s}_\vee \Rightarrow x^2_\vee > 0$$

1D and 4D future are the first two entries of the symmetric space chain $\mathbf{GL}(\mathbb{C}^n)/\mathbf{U}(n)$, $n = 1, 2, \ldots$, which are the manifolds of the unitary groups in the general linear group, canonically parametrized in the polar decomposition $g = u \circ |g|$ with the real $n^2$-dimensional ordered absolute values $x_\vee = |g| = \sqrt{g^* \circ g} \in$

$\mathbb{R}^{n^2}_{\vee}$ of the general linear group. They are the positive cone of the ordered $C^*$-algebras with the complex $n \times n$ matrices.

In residual representations the future cone $\mathbb{R}^{1+s}_{\vee} = G/H$ is canonically parametrized by translations which constitute the tangent space $\log G/H$ of the future cone

$$\mathbb{R}^{1+s} \cong \begin{cases} \log \mathbf{D}(1), & s = 0 \\ \log \mathbf{D}(1) \oplus \log \mathbf{SO}_0(1,1), & s = 1 \\ \log \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2), & s = 3 \end{cases}$$

The cone is embedded into its tangent space. The future cone $\mathbb{R}^4_{\vee} \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ as the orientation manifold of unitary groups is taken as model for nonlinear spacetime. The $\mathbf{GL}(\mathbb{C}^2)$-action by left multiplication involves the external Lorentz group. The group $\mathbf{U}(2)$ of the equivalence classes is used for internal degrees of freedom (hyperisospin). The related structures (Saller, 1998, 2001b) will not be considered in more detail in the following.

### 10.2. Residual Representations of Time Future

The residual representations of time future by the advanced energy distributions are characterized by one compact invariant and nildimension $N$

$$\frac{1}{2i\pi} \frac{\Gamma(1+N)}{(q - io - m)^{1+N}} = \frac{1}{2}\left[\delta^{(N)}(m-q) + \frac{1}{i\pi}\frac{\Gamma(1+N)}{(qP - m)^{1+N}}\right]$$

They are representation matrix elements of the causal group $\mathbf{D}(1)$

$$\mathbb{R}_{\vee} \ni t_{\vee} \mapsto \int \frac{dq}{2i\pi} \frac{\Gamma(1+N)}{(q - io - m)^{1+N}} e^{iqt} = (it_{\vee})^N e^{imtv}$$

### 10.3. Residual Representations of 2-D Future

The residual representations of 2D future will be constructed from the advanced energy–momentum distributions

$$\frac{1}{2i\pi}\frac{1}{(q - io)^2 - m^2} = \frac{1}{2}\left[\in(q_0)\delta(q^2 - m^2) + \frac{1}{i\pi}\frac{1}{q_{\mathrm{P}}^2 - m^2}\right]$$

With the Fourier transforms and their partial projections one obtains for the representations of time future and position

$$\begin{aligned} \pi g(m^2, x) &= \int d^2 q e^{iqx} \tilde{g}(m^2, q) = \int dq_3\, e^{-iq_3 x^3} g(q_0, x^0) \\ &= \int dq_0\, e^{iq_0 x^0} \left[\vartheta\left(q_0^2 - m^2\right) g^c(q_3, x^3) + \vartheta\left(m^2 - q_0^2\right) g^{nc}(iq_3, x^3)\right] \end{aligned}$$

Representations of Spacetime Future $\mathbb{R}^2_\vee \cong \mathbb{R}_\vee \oplus \mathbb{R}$

| | Spacetime future $(x_\vee = \vartheta(x^0)\vartheta(x^2)x)$ | Time frame $(x^0_\vee = t_\vee)$ | Position frame $(x^3 = z)$ |
|---|---|---|---|
| $\tilde{g}(m^2, q)$ | $g(m^2, x)$ | $g(\lvert m\rvert, t_\vee) \in \mathbf{D(1)}$ $(q_0, Q_3) = (\lvert m\rvert, 0)$ | $g^{nc}(i\lvert m\rvert, z) \in \mathbf{SO}_0(1,1)$ $(q_0, q_3) = (0, i\lvert m\rvert)$ |
| $\frac{1}{2\pi}\frac{1}{(q-io)^2-m^2}$ | $\mathcal{E}_0\left(\frac{m^2x^2_\vee}{4}\right)$ | $-\frac{\sin\lvert m\rvert t_\vee}{\lvert m\rvert}$ | $\frac{e-\lvert mz\rvert}{\lvert m\rvert}$ |
| $\frac{1}{2i\pi}\frac{q}{(q-io)^2-m^2}$ | $\bullet\ \partial_\vee\mathcal{E}_0\left(\frac{m^2x^2_\vee}{4}\right)$ | $\cos mt_\vee$ | $e(z)e^{-\lvert mz\rvert}$ |
| $\frac{1}{2i\pi}\frac{q}{((q-io)^2-m^2)^2}$ | $\frac{xv}{2}\mathcal{E}_0\left(\frac{m^2x^2_\vee}{4}\right)$ | $-\frac{tv\sin\lvert m\rvert tv}{2\lvert m\rvert}$ | $-z\frac{e^{-\lvert mz\rvert}}{2\lvert m\rvert}$ |

$$\frac{\partial}{\partial\frac{x^2}{4}}\mathcal{E}_0\left(\frac{m^2x^2_\vee}{4}\right) = \delta\left(\frac{x^2_\vee}{4}\right) - m^2\mathcal{E}_1\left(\frac{m^2x^2_\vee}{4}\right) \quad \text{with} \quad \delta\big(x^2_\vee\big) = \vartheta(x_0)\delta(x^2)$$

$$\frac{\partial}{\partial m^2}\frac{\partial}{\partial\frac{x^2}{4}}\mathcal{E}_0\left(\frac{m^2x^2_\vee}{4}\right) = -\mathcal{E}_0\left(\frac{m^2x^2_\vee}{4}\right), \quad \partial_\vee = \frac{\partial}{\partial x\vee} = 2xv\frac{\partial}{\partial x^2}$$

With $t_\vee = \vartheta(t)t = \frac{1+\epsilon(t)}{2}t$ the time future projections, i.e., the representation matrix elements of the causal group $\mathbf{D}(1)$, are combined from Dirac and principal value contribution. The position space projections, i.e., representation matrix elements of the orthochronous group $\mathbf{SO}_0(1, 1)$, come from the principal value only.

Spacetime future representation matrix elements have to be functions, i.e., the Dirac distribution $\delta(x^2_\vee)$ on the forward lightcone in the Lorentz vector gives no representations, marked by $\bullet$. The future lightlike translations $\frac{1_2\pm\sigma_3}{2}x^0_\vee$ are no elements of strict future $x^2_\vee > 0$.

Two-dimensional future is the *rank 2 real Lie group* $\mathbf{D}(1) \times \mathbf{SO}_0(1,1)$. The residual representations of these two noncompact groups will be characterized by two invariants for the characters, both from a continuous spectrum. Therefore, the dipole in the residual representation will be supported by *two Lorentz invariants* for the hyperbolic-spherical singularity surface with the pole function

$$\frac{1}{q^2 - m_0^2} - \frac{1}{q^2 - m_3^2} = \frac{m_0^2 - m_3^2}{\big(q^2 - m_0^2\big)\big(q^2 - m_3^2\big)} = \int_{m_3^2}^{m_0^2} dm^2 \frac{1}{(q^2 - m^2)^2}$$

By the Lorentz compatible embedding with tangent $\mathbb{R}^2$-translations and energy–momenta both invariants contribute to representations of the time group $\mathbf{D}(1)$ and the position space $\mathbf{SO}_0(1, 1)$.

On the lightcone $x^2 = 0$, where time and position translations coincide $x^3 = \pm x^0$, the contributions from both invariants cancel each other as seen for the vector representation

$$\text{Spacetime future:} \mathbb{R}^2_\vee \ni x_\vee \mapsto \int_{m_3^2}^{m_0^2} dm^2 \int \frac{d^2q}{2i\pi} \frac{q}{((q-io)^2 - m^2)^2} e^{iqx}$$
$$= -\frac{x\nu}{2}\pi \left[ m_0^2 \mathcal{E}_1 \left( \frac{m_0^2 x^2 \nu}{4} \right) - m_3^2 \mathcal{E}_1 \left( \frac{m_3^2 x_\vee^2}{4} \right) \right]$$

with the projection $x_\vee = t_\vee \mathbf{1}_2 + z\sigma_3$ on time future and position

$$\text{Time future:} \mathbb{R}_\vee \ni t_\vee \mapsto \int_{m_3^2}^{m_0^2} dm^2 \int \frac{dq}{2i\pi} \frac{q}{((q-io)^2 - m^2)^2} e^{iqt}$$
$$= \cos m_0 t_\vee - \cos m_3 t_\vee$$
$$\text{Position:} \mathbb{R} \ni z \mapsto \int_{m_3^2}^{m_0^2} dm^2 \int \frac{dq}{2i\pi} \frac{q}{(q^2 + m^2)^2} e^{-iqz}$$
$$= \epsilon(z) \frac{e^{-|m_0 z|} - e^{-|m_3 z|}}{2}$$

The energy projected trace residues of the representation functions are

$$\operatorname{tr} \operatorname*{Res}_{\mu} \frac{m_0^2 - m_3^2}{(q^2 - m_0^2)(q^2 - m_3^2)} = \begin{cases} 1, & \mu^2 = m_0^2 \\ -1, & \mu^2 = m_3^2 \end{cases}$$

## 10.4. Residual Representations of 4D Future

Two-dimensional future is a Cartan subgroup of 4D future with additional 2-sphere degrees of freedom $\mathbb{R}^4_\vee / \mathbb{R}^2_\vee \cong \Omega^2$.

The residual representations of 4D future by advanced energy–momentum distributions have as projections to time future and position

$$\pi g(m^2, x) = \int \frac{d^4q}{\pi} e^{iqx} \tilde{g}(m^2, q) = \int \frac{d^3q}{\pi} e^{-i\vec{q}\vec{x}} g(q_0, x^0)$$
$$= \int dq_0 \, e^{iq_0 x^0} \left[ \vartheta(q_0^2 - m^2) g^c(|\vec{q}|, \vec{x}) + \vartheta(m^2 - q_0^2) g^{nc}(i|\vec{q}|, \vec{x}) \right]$$

$$\left( \frac{\partial}{\partial \frac{x^2}{4}} \right)^2 \mathcal{E}_0 \left( \frac{m^2 x_\vee^2}{4} \right) = \delta' \left( \frac{x_\vee^2}{4} \right) - m^2 \delta \left( \frac{x_\vee^2}{4} \right) + m^4 \mathcal{E}_2 \left( \frac{m^2 x_\vee^2}{4} \right)$$

Four-dimensional future is the real homogeneous space $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ with rank 2 for a Cartan subgroup $\mathbf{D}(1) \times \mathbf{SO}_0(1, 1)$. Therefore, its residual representation will be supported by two invariants as for the 2-dimensional case with a

Representations of Spacetime Future $\mathbb{R}^4_\vee \cong \mathbb{R}_\vee \oplus \mathbb{R}^3$

| | Spacetime future $(x_\vee = \vartheta(x^0)\vartheta(x^2)x)$ | Time frame $(x^0_\vee = t_\vee)$ | Position frame |
|---|---|---|---|
| $\tilde{g}(m^2, q)$ | $g(m^2, x)$ | $g(\vert m\vert, t_\vee) \in \mathbf{D(1)}$ $(q_0, \vec{q}) = (\vert m\vert, 0)$ | $g^{nc}(i\vert m\vert, \vec{x}) \in \mathbf{SO}_0(1{,}3)/\mathbf{SO}(3)$ $(q_0, \vert\vec{q}\vert) = (0, i\vert m\vert)$ |
| $\frac{1}{\pi}\frac{1}{(q-io)^2-m^2}$ | $\bullet -\frac{\partial}{\partial \frac{x^2}{4}}\mathcal{E}_0\left(\frac{m^2x_\vee^2}{4}\right)$ | $-\frac{\sin\vert m\vert t_\vee}{\vert m\vert}$ | $-\frac{e-\vert m\vert r}{r}$ |
| $\frac{1}{\pi}\frac{1}{((q-io)^2-m^2)^2}$ | $\mathcal{E}_0\left(\frac{m^2x_\vee^2}{4}\right)$ | $-\frac{\sin\vert m\vert t_\vee - \vert m\vert t_\vee \cos m t_\vee}{2\vert m\vert^3}$ | $\frac{e-\vert m\vert r}{2\vert m\vert}$ |
| $\frac{1}{i\pi}\frac{q}{(q-io)^2-m^2}$ | $\bullet\partial_\vee\frac{\partial}{\partial \frac{x^2}{4}}\mathcal{E}_0\left(\frac{m^2x_\vee^2}{4}\right)$ | $\cos m t_\vee$ | $\frac{\vec{x}}{r}\frac{1+\vert m\vert r}{r^2}e^{-\vert m\vert r}$ |
| $\frac{1}{i\pi}\frac{q}{((q-io)^2-m^2)}^2$ | $\bullet\partial_\vee\mathcal{E}_0\left(\frac{m^2x_\vee^2}{4}\right)$ | $\frac{t_\vee \sin\vert m\vert t_\vee}{2\vert m\vert}$ | $\frac{\vec{x}}{r}\frac{e^{-\vert m\vert r}}{2}$ |
| $\frac{1}{i\pi}\frac{2q}{((q-io)^2-m^2)^3}$ | $-\frac{xv}{2}\mathcal{E}_0\left(\frac{m^2x_\vee^2}{4}\right)$ | $-\frac{t_\vee(\sin\vert m\vert t_\vee - \vert m\vert t_\vee \cos m t_\vee)}{4\vert m\vert^3}$ | $-\vec{x}\frac{e^{-\vert m\vert r}}{2\vert m\vert}$ |

characteristic additional dipole structure (Heisenberg, 1967) to take into account the 2-sphere degrees of freedom in 3D position

$$\frac{1}{q^2-m_0^2} - \frac{1}{q^2-m_3^2} - \frac{m_0^2-m_3^2}{\left(q^2-m_3^2\right)^2} = \frac{\left(m_0^2-m_3^2\right)^2}{\left(q^2-m_0^2\right)\left(q^2-m_3^2\right)^2}$$
$$= \int_{m_3^2}^{m_0^2} dm^2\left(m_0^2-m^2\right)\frac{2}{(q^2-m^2)^3}$$

Again, both invariants contribute to representations of the time group $\mathbf{D}(1)$ and the symmetric position space $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,1) \times \Omega^2 \cong \mathbb{R}^3$.

There is one aditional noncompact continuous invariant compared with the one compact mass invariant $m^2$ of the Poincaré group for free particles as used in the Wigner classification

| Rank | Lorentz | Poincaré | Expansion |
|---|---|---|---|
| 1 | $\mathbf{SO}_0(1{,}1)$ | $\mathbf{SO}_0(1,1)\,\vec{\times}\,\mathbb{R}^2$ | $\mathbf{SO}_0(2,1)$ |
| 2 | $\mathbf{SO}_0(1{,}3)$ | $\mathbf{SO}_0(1,3)\,\vec{\times}\,\mathbb{R}^4$ | $\mathbf{SO}_0(2,3)$ |

With two invariants the vector representations of 4-dimensional future are

$$\text{Spacetime future:}\ \mathbb{R}^4_\vee \ni x_\vee \mapsto \int_{m_3^2}^{m_0^2} dm^2\left(m_0^2-m^2\right)\int\frac{d^4q}{2i\pi^2}\frac{2q}{((q-io)^2-m^2)^3}e^{iqx}$$
$$= \frac{x\vee}{2}\pi\left[m_0^4\mathcal{E}_2\left(\frac{m_0^2x^2v}{4}\right) - m_3^4\mathcal{E}_2\left(\frac{m_3^2x_\vee^2}{4}\right)\right.$$

$$+\left(m_0^2-m_2^2\right)m_3^2\mathcal{E}_1\left(\frac{m_3^2x_\vee^2}{4}\right)\Bigg]$$

with the projection $x_\vee = t_\vee \mathbf{1}_2 + \vec{x}$ on time future and 3-dimensional position

$$\text{Time future:}\ \mathbb{R}_\vee \ni t_\vee \mapsto \int_{m_3^2}^{m_0^2} dm^2\left(m_0^2-m^2\right)\int \frac{dq}{2i\pi}\frac{q}{((q-io)^2-m^2)^3}e^{iqt}$$

$$= \cos m_0 t_\vee - \cos m_3 t_\vee + \left(m_0^2-m_3^2\right)\frac{t_\vee \sin m_3 t_\vee}{2m_3}$$

$$\text{Position:}\ \mathbb{R}^3 \ni \vec{x} \mapsto \int_{m_3^2}^{m_0^2} dm^2\left(m_0^2-m^2\right)\int \frac{d^3q}{2i\pi^2}\frac{i\vec{q}}{(q^2+m^2)^3}e^{-i\vec{q}\vec{x}}$$

$$= \frac{\vec{x}}{r}\left[\frac{1+|m_0|r}{r^2}e^{-|m_0|r} - \frac{1+|m_3|r}{r^2}e^{-|m_3|r} + \left(m_0^2-m_3^2\right)\frac{e-|m_3|r}{2}\right]$$

The energy projected trace residues of the representation functions are as for the Cartan substructure

$$\mathrm{tr}\,\underset{\mu}{\mathrm{Res}}\frac{\left(m_0^2-m_3^2\right)^2}{\left(q^2-m_0^2\right)\left(q^2-m_3^2\right)^2} = \underset{\mu}{\mathrm{Res}}\left[\frac{2q^3}{q^2-m_0^2}-\frac{2q^3}{q^2-m_3^2}\right] = \begin{cases} 1, & \mu^2=m_0^2 \\ -1, & \mu^2=m_3^2\end{cases}$$

A simple pole $\frac{q}{q^2-m^2}$ has a positive energy projected residue, its mass can be associated to a particle. The related irreducible time translation representation with positive normalization in an associate inner product space can be taken over to define a Feynman propagator as Fourier transformation of $q\delta(q^2-m^2)$ with unitary representations $e^{iqx}$ of spacetime translations by a free particle. Dipoles $\frac{q}{(q^2-m^2)^2}$ cannot be related to probability valued eigenvectors for translations, they come from nondecomposable 2-dimensional nondiagonalizable translation representations with triagonal nilpotent Jordan contributions and with a ghost metric (Saller, 1999a,b). Product representations with a dipole can involve poles for particles. A dipole $\frac{1}{(q^2-m^2)^2}$ has a nontrivial momentum projected residue.

## 11. MATTER AS SPACETIME SPECTRUM

### 11.1. Residual Representations of Tangent Spaces

Complex pole functions of the translation characters (energy–momenta) $q \mapsto \frac{Q(q)}{P(q)}$ can be used both for the representations of a symmetric space (spacetime) and for the representations of its tangent space (spacetime translations).

On a symmetric space function $(G/H)_{\text{repr}} \ni x \mapsto g(x)$ with canonical parametrization, e.g., $x^N e^{imx}$ for $\mathbf{D}(1)$ or $e^{i|m|\vec{x}}$ for $\mathbf{SU}(2)$, the tangent space (Lie algebra) action involves the corresponding derivatives, e.g.

$$\frac{d}{dx} \text{for} \log \mathbf{D}(1) \cong \mathbb{R}, \quad \frac{\partial}{\partial \vec{x}}, \frac{\partial^2}{\partial \vec{x}^2} \text{ for} \log \mathbf{SO}_0(1,3)/\log \mathbf{SO}(3) \cong \mathbb{R}^3$$

Therewith a *tangent distribution* of a symmetric space, e.g., a Lie algebra distribution for a Lie group, will be defined by an inverse derivative with an invariant pole and a residue $a_1$, familiar as *Green distributions of differential equations* (in general no functions). Its Fourier transform defines a complex tangent representation function. Tangent distributions come with different integration contours. In contrast to the normalization of Cartan group representations by the group unit, the residue of a tangent representation has to be determined by another structure (below).

The causal group time is isomorphic to its tangent space. Therefore the tangent representation functions with appropriate residue are also group representation functions

$$\text{time } \mathbf{D}(1) \cong \mathbb{R} : \frac{a_1}{q - m}$$

For 3D position with the rank 2 Euclidean semidirect group there are two types of tangent functions—for integer and half integer spin

$$\text{position } \mathbf{SO}(3) \vec{\times} \mathbb{R}^3, \quad \mu^2 = \pm m^2 : \begin{cases} J = 0, 1, \ldots : \frac{a-1(\vec{q})^{2J}}{(\vec{q}^2-\mu^2)^{I+J}}, & \text{e.g., } \frac{a-1}{\vec{q}^2-\mu^2} \\ J = \frac{1}{2}, \frac{3}{2}, \ldots : \frac{a-1(\vec{q})^{2J}}{(\vec{q}^2-\mu^2)^{\frac{1}{2}+J}}, & \text{e.g., } \frac{a-1\vec{q}}{\vec{q}^2-\mu^2} \end{cases}$$

The Fourier transforms involve the Yukawa potential and force.

The tangent functions for time and position have to be embedded into Minkowski spacetime tangent function: For 2-dimensional spacetime one has with the rank 1 Poincaré group

$$\text{Spacetime } \mathbf{SO}_0(1,1) \vec{\times} \mathbb{R}^2 : \frac{a-1}{q^2 - m^2}, \quad \frac{a-1q}{q^2-m^2}$$

For 4-dimensional spacetime with the rank 2 Poincaré group there are two tangent function types

$$\text{Spacetime } \mathbf{SO}_0(1,3) \vec{\times} \mathbb{R}^4 : \begin{cases} J = 0, 1, \ldots : \frac{a-1(q)^{4L}_{2J}}{(q^2-m^2)^{1+2L}} & \text{with } 2L \geq J \\ J = \frac{1}{2}, \frac{3}{2}, \ldots : \frac{a-1(q)^{1+4L}_{2J}}{(q^2-m^2)^{1+2L}} & \text{with } 2L \geq J - \frac{1}{2} \\ \qquad \text{e.g., } \frac{a-1q}{q^2-m^2} \end{cases}$$

For a given $J$ there are different embeddings $L$, as discussed above.

There is a decisive difference of tangent distributions of position $\mathbb{R}^3$ and 2D and 4D spacetime $\mathbb{R}^{2,4}$ compared with those of time $\mathbb{R}$. In general, tangent distributions $l \in \mathcal{G}'$ are no symmetric space functions $g \in \mathcal{G}$, i.e., $\mathcal{G}' \supseteq \mathcal{G}$. They are derivatives thereof with respect to the canonical parameters, e.g., $\frac{e^{-\mu r}}{2r} = -\frac{d}{dr^2}\frac{e^{-\mu r}}{\mu}$ where the Yukawa potential arise from the tangent representation functions $\{\frac{1}{(\vec{q}^2+\mu^2)^{1+N}} | N = 0, 1, \ldots\}$ and the exponential from the symmetric space representation functions $\{\frac{1}{(\vec{q}^2+\mu^2)^{2+N}} | N = 0, 1, \ldots\}$. In general, the tangent representation functions constitute a vector space only. In contrast to the pointwise multiplicative property of symmetric space functions $\mathcal{G} \bullet \mathcal{G} \to \mathcal{G}$ and convolution for their Fourier transforms $\tilde{\mathcal{G}} * \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ the requirement of multiplicative stability for the tangent distributions does not make sense ("divergencies"). Translations and their representations can be added, but, in general, they cannot be multiplied. For example, a squared Yukawa potential $\frac{e^{-2|m|r}}{r^2}$ does not make sense as a representation. Or, Lie algebra representation matrix elements have no associative multiplicative structure. However, a tangent vector space (Lie algebra) should be a module $\tilde{\mathcal{G}}' \in \text{mod}_{\tilde{\mathcal{G}}}$ with respect to the residual action with functions $\tilde{\mathcal{G}}$ for symmetric space (group) representations, i.e., $\tilde{\mathcal{G}} * \tilde{\mathcal{G}}' \to \tilde{\mathcal{G}}'$ and for the Fourier transforms $\mathcal{G} \bullet \mathcal{G}' \to \mathcal{G}'$

$$\begin{aligned}
&\text{symmetric space} * \text{symmetric space} \to \text{symmetric space}\\
&\text{symmetric space} * \text{tangent space} \quad\;\; \to \text{tangent space}
\end{aligned}$$

For example, a Lie group $G$ acts adjointly $G \times G' \to G'$, $\text{Ad}\, g(l) = glg^{-1}$, upon its Lie algebra $G' = \log G$ or on its tensor fields.

With the tangent distributions dual to the symmetric space functions the residual product (convolution) of a tangent space function with a group function arises in the dual product

$$\begin{aligned}
\mathcal{G}' \times \mathcal{G} \to \mathbb{C},\ \langle l, g \rangle &= \int l(x) dx\, g(x)\\
&= \int d^d x \int d^d q e^{iqx} (\tilde{l} * \tilde{g})(q)\\
&= (2\pi)^d (\tilde{l} * \tilde{g})(0)
\end{aligned}$$

With $\langle l, g \rangle = 1$ the tangent and symmetric space functions are called dual to each other.

## 11.2. Eigenvalue Equations

The tangent action defines eigenfunctions for an invariant $\mu \in \mathbb{C}$ by, e.g.

$$\frac{1}{\mu}\frac{d}{dx} g(x) = g(x), \quad \frac{1}{\mu^2}\frac{\partial^2}{\partial \vec{x}^2} g(\vec{x}) = g(\vec{x}), \quad \frac{1}{\mu}\frac{\partial}{\partial \vec{x}} g(\vec{x}) = g(\vec{x})$$

The invariant is the solution of the *eigenvalue equation* for the massless tangent function $l_0$, $\delta' * l_0 = \delta$, (inverse derivative), e.g., $\tilde{l}_0(q) = \frac{\mu}{q}, \frac{\mu\vec{q}}{\vec{q}^2}, \frac{\mu^2}{\vec{q}^2}, \frac{mq}{q^2}$, with the unit or the Lorentz compatibly embedded unit on the r.h.s.—with the examples

$$\begin{aligned}
\text{Time}\,\mathbb{R}: &\quad \frac{m}{q} = 1 \Rightarrow q = m\\
\text{Position}\,\mathbb{R}^3: &\quad \frac{\mu\vec{q}}{\vec{q}^2} = \mathbf{1}_2, \quad \frac{\mu^2}{\vec{q}^2} = 1 \Rightarrow \vec{q}^2 = \mu^2\\
\text{Spacetime}\,\mathbb{R}^{1+s}: &\quad \frac{mq}{q^2} = \mathbf{1}_2, \quad \frac{m^2}{q^2}\left(\mathbf{1}_{1+s} - \frac{q \otimes \check{q}}{q^2}\right) = \left(\mathbf{1}_{1+s} - \frac{q \otimes \check{q}}{q^2}\right) \Rightarrow q^2 = m^2
\end{aligned}$$

To obtain invariants for a product representation a function for a symmetric space representation acts by residual product upon the massless tangent function leading to another tangent function

$$\tilde{l}_0 : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}', \quad \tilde{g} \mapsto \tilde{l}_0 * \tilde{g}$$

with the invariant arising from the eigenvalue equation

$$\tilde{l}_0 * \tilde{g}(q) = 1 \Rightarrow q = \mu$$

This amounts to a normalization of the $q$-dependent residue arising in a convolution.

For example, the residual action of the tangent function $\frac{m}{q}$ of abelian time $\mathbf{D}(1)$ on an irreducible representation $\frac{1}{q-M}$ gives $m + M$ as eigenvalue for the product representation

$$\begin{aligned}
\frac{m}{q} : \mathbb{R}_\vee \to \mathbb{R} : \frac{1}{q-M} \mapsto \frac{m}{q} \overset{\mathrm{R}}{*} \frac{1}{q-M} &= \frac{m}{q-M}\\
\frac{m}{q-M} &= 1 \Rightarrow q = M + m
\end{aligned}$$

## 11.3. Eigenvalues for Position Bound Waves

The Hamiltonian for the nonrelativistic hydrogen atom involves the Kepler potential that is a tangent distribution arising by Fourier transformation of a massless function representing the position translations $\mathbb{R}^3$

$$H = \frac{\vec{p}^2}{2} - \frac{1}{r_P}, \quad \frac{1}{r_P} = \int \frac{d^3q}{2\pi^2}\frac{1}{\vec{q}^2}e^{-i\vec{q}\vec{x}}$$

The eigenvalue equation involves the residual product with the wave functions $g$ as position representation matrix elements

$$Hg(\vec{x}) = Eg(\vec{x}) \iff \left[\frac{\vec{q}^2}{2} - \frac{i}{\vec{q}^2} \overset{\mathrm{R}}{*}\right] \tilde{g}(\vec{q}) = E\tilde{g}(\vec{q}) \text{ with } \overset{\mathrm{R}}{*} = \frac{*}{2i\pi^2}$$

The residual product of the massless tangent representation function $\frac{i}{\vec{q}^2}$ with the position representation functions $\tilde{g}$ with invariant $\mu \in (|m|, \mp i(|m| \pm io))$ gives tangent representation functions, e.g., for scalar representations

$$\frac{i}{\vec{q}^2} \overset{\mathrm{R}}{*} \tilde{g}(\vec{q}) = \tilde{l}(\vec{q}) \quad \text{with} \quad \begin{cases} \tilde{g}(\vec{q}) \in \left\{ \sum_{n=0}^{N} \frac{a_{-2-n}}{(\vec{q}^2+\mu^2)^{2+n}} \; | a_{-2-n} \in \mathbb{C} \right\} \\ \tilde{l}(\vec{q}) \; \in \left\{ \sum_{n=1}^{N} \frac{a_{-1-n}}{(\vec{q}^2+\mu^2)^{1+n}} \; | a_{-1-n} \in \mathbb{C} \right\} \end{cases}$$

$$\text{e.g., for} \quad g(x) = \int \frac{d^3q}{2\pi^2} \frac{2\mu}{(\vec{q}^2 + \mu^2)^2} e^{-i\vec{q}\vec{x}} = e^{-\mu r}$$

$$\Rightarrow \frac{i}{\vec{q}^2} \overset{\mathrm{R}}{*} \frac{2\mu}{(\vec{q}^2 + \mu^2)^2} = \frac{1}{\vec{q}^2 + \mu^2}$$

Therewith the eigenvalue problem can be solved by noncompact position representation functions (Hilbert space bound waves), e.g., by the irreducible scalar position representation for the ground state $|1, \vec{0}\rangle \sim e^{-r}$

$$\left[\frac{\vec{q}^2}{2} - \frac{i}{\vec{q}^2} \overset{\mathrm{R}}{*}\right] \frac{1}{(\vec{q}^2 + \mu^2)^2} = E \frac{1}{(\vec{q}^2 + \mu^2)^2} \iff \frac{\vec{q}^2}{2} - \frac{\vec{q}^2 + \mu^2}{2\mu}$$

$$= E \Rightarrow \begin{cases} E = -\frac{\mu^2}{2} \\ \mu = 1 \end{cases}$$

Nontrivial knots $N = 1, 2, \ldots$ lead to the Laguerre polynomials as linear combinations of position representation functions. Analogously, harmonic polynomials for angular momenta $L = 1, 2, \ldots$ can be included.

## 11.4. The Invariant Mass Ratio for Spacetime

In general and in contrast to residual product stable energy and momentum pole functions, $\mathcal{P}(\overline{\mathbb{C}}) \overset{\mathrm{R}}{*} \mathcal{P}(\overline{\mathbb{C}}) \to \mathcal{P}(\overline{\mathbb{C}})$, the residual products of energy–momentum $q^2$-pole functions for representations of rank 2 nonlinear spacetime $\mathbb{R}_{\vee}^{1+s}$ with hyperbolic-spherical singularity surfaces $q^2 = m^2$ do not produce rational complex functions with $q^2$-poles which would determine the invariants of product representations. The $q^2$-dependent residue of the convolution gives integrals over rational functions, e.g.

$$\text{Spacetime: } \int_0^1 d\zeta \, \frac{1}{q^2\zeta - m^2} = \frac{\log\left(1 - \frac{q^2}{m^2}\right)}{q^2}$$

In the following an attempt is made to determine invariant masses and normalizations of energy–momentum poles for the representations of the time translations $\mathbb{R}$, Lorentz compatibly embedded into spacetime translations $\mathbb{R}^{1+s}$. Perhaps, one can characterize this as an attempt to find a Lorentz compatible solution of the bound state problem in the potential $V_3(r)$ as given above in the projection of the vector representation of nonlinear spacetime $\mathbb{R}^4_\vee$ to the homogeneous position space $\mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \mathbb{R}^3$. The superposition of Yukawa and exponential potentials

$$\vec{F}(\vec{x}) = -\frac{\partial}{\partial \vec{x}} V_3(r), \quad V_3(r) = \int_{m_3^2}^{m_0^2} dm^2 \big(m_0^2 - m^2\big) \int \frac{d^3q}{m\pi^2} \frac{1}{(\vec{q}^2 + m^2)^3} e^{-i\vec{q}\vec{x}}$$
$$= \frac{e^{-|m0|r} - e^{-|m_3|r}}{r} + \frac{m_0^2 - m_3^2}{2|m_3|} e^{-|m_3|r}$$

is the 2-sphere spread of a noncompact representation of 1D position with a $z$-proportional contribution from the dipole (nildimension $N = 1$)

$$V_3(r) = -\frac{d}{dr^2} V_1(r), \quad V_1(z) = \int_{m_3^2}^{m_0^2} dm^2 \big(m_0^2 - m^2\big) \int \frac{dq}{\pi} \frac{2}{(\vec{q}^2 + m^2)^3} e^{-iqz}$$
$$= \int_{m_3^2}^{m_0^2} dm^2 \big(m_0^2 - m^2\big) \left(\frac{d}{dm^2}\right)^2 \frac{e^{-|mz|}}{|m|}$$
$$= 2\frac{e^{-|m_0 z|}}{|m_0|} - \left[2 - \frac{m_0^2 - m_3^2}{m_3^2}(1 + |m_3 z|)\right] \frac{e^{-|m_3 z|}}{|m_3|}$$

The residual product $\frac{q}{q^2} \overset{\mathrm{R}}{*} \tilde{g}$ of the massless vector function for a spacetime tangent representation with the spacetime vector representation function $\tilde{g}$, characterized by two invariants gives

$$\frac{q}{q^2} : \mathbb{R}^2_\vee \to \mathbb{R}^2 \; : \; \frac{q}{q^2} \overset{\mathrm{R}}{*} \int_{m_3^2}^{m_0^2} dm^2 \frac{q}{(q^2 - m^2)^2}$$
$$= -\left(\frac{1}{2}\mathbf{1}_2 + q \otimes \check{q} \frac{\partial}{\partial q^2}\right) \int_{m_3^2}^{m_0^2} dm^2 \int_0^1 d\zeta \, \frac{1}{q^2\zeta - m^2}$$

$$\frac{q}{q^2} : \mathbb{R}^4_\vee \to \mathbb{R}^4 \ : \ \frac{q}{q^2} \overset{\mathrm{R}}{*} \int_{m_3^2}^{m_0^2} dm^2 \big(m_0^2 - m^2\big) \frac{2q}{(q^2-m^2)^3}$$
$$= -\left(\frac{1}{2}\mathbf{1}_4 + q \otimes \check{q} \frac{\partial}{\partial q^2}\right) \int_{m_3^2}^{m_0^2} dm^2 \big(m_0^2 - m^2\big) \int_0^1 d\zeta \, \frac{1-\zeta}{q^2\zeta - m^2}$$

The massless tangent function has a hyperbolic singularity surface. With at least one nontrivial invariant, the spacetime representation function has a hyperbolic-spherical singularity surface. Therefore, invariants on the hyperbolic surface are combined with invariants on hyperbolic and spherical surfaces. There is no combination of invariants that are both on spherical surfaces.

The invariant $m_0^2 \neq 0$ for the normalized embedded time representation $\frac{q}{q^2-m_0^2}$ is used as unit

$$m_0^2 \cong 1, \quad \frac{q}{|m_0|} \cong q, \quad \frac{m_3^2}{m_0^2} \cong m_3^2$$

The eigenvalue functions are the $q^2$-dependent residues

$$\tilde{l}_{\mathbf{1}}(q^2) = \frac{M^2(\frac{1}{q^2})}{q^2} = \begin{cases} -\int_{m_3^2}^1 \frac{dm^2}{2} & \int_0^1 d\zeta \, \frac{1}{q^2\zeta - m^2} \quad \text{for} \quad \mathbb{R}^2 \\ -\int_{m_3^2}^1 \frac{dm^2(1-m^2)}{2} & \int_0^1 d\zeta \, \frac{1-\zeta}{q^2\zeta - m^2} \quad \text{for} \quad \mathbb{R}^4 \end{cases}$$

The residual product will be used to establish duality between spacetime and tangent representation in the normalization $(\frac{q}{q^2} \overset{\mathrm{R}}{*} \tilde{g})(0) = \mathbf{1}_{1+s}$. This duality condition requires an eigenvalue at mass $q^2 = 0$, i.e., $\frac{\vec{q}^2}{q_0^2} = 1$, and determines the ratio $\frac{m_3^2}{m_0^2}$ of the invariants for rank 2 spacetime

$$1 = \tilde{l}_{\mathbf{1}}(0) = \begin{cases} -\frac{\log m_3^2}{2} & \Rightarrow \frac{m_3^2}{m_0^2} = e^{-2} \sim \frac{1}{7.4} \quad \text{for} \quad \mathbb{R}^2 \\ -\frac{\log m_3^2 + 1 - m_3^2}{4} & \Rightarrow \frac{m_3^2}{m_0^2} \sim e^{-5} \sim \frac{1}{148.4} \quad \text{for} \quad \mathbb{R}^4 \end{cases}$$

## 12. RESIDUES OF TANGENT REPRESENTATIONS

### 12.1. Geometric Transformation and Mittag–Leffler Sum

The exponential from the Lie algebra $\mathbb{R}$ (time translations) to the group exp $\mathbb{R} = \mathbf{D}(1)$ can be reformulated in the language of residual representations with energy functions by a *geometric series*

$$e^{imt} = \oint \frac{dq}{2i\pi} \frac{1}{q-m} e^{iqt}$$
$$= \sum_{k=0}^{\infty} \frac{(imt)^k}{k!} = \oint \frac{dq}{2i\pi} \frac{1}{q} \sum_{k=0}^{\infty} \frac{m^k}{q^k} e^{iqt}$$

The transformations involved

$$z \mapsto \frac{1}{z} = w \mapsto \frac{w}{1-w} = \frac{1}{z-1}, \quad \text{e.g., } z = \frac{q}{m}, \frac{q^2}{m^2}$$

are elements of the broken rational (conformal) bijective transformations of the closed complex plane

$$\overline{\mathbb{C}} \ni z \mapsto \frac{az+\beta}{yz+\beta} \in \overline{\mathbb{C}}$$

with real coefficients as group isomorphic to

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{SL}(\mathbb{R}^2) \sim \mathbf{SU}(1,1) \sim \mathbf{SO}(1,2)$$

For det $A = 1$upper and lower half plane $x \pm io$ remain stable. The eigenvalue $w = z = 1$ becomes a pole

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : w \mapsto \frac{w}{-w+1}, \quad 1 \mapsto \infty, \quad 0 \leftrightarrow 0$$

With one fixpoint $w = 0$ the transformation is parabolic, i.e., an element of the $\mathbb{R}$-isomorphic subgroup $\left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$.

The geometric transformation will be generalized to associate pole functions to the complex eigenvalue functions for spacetime with $z = \frac{q^2}{m^2}$

$$z \mapsto l(z) \mapsto \frac{l(z)}{1-l(z)}$$

An eigenvalue, i.e., a zero of the denumerator $z_0 \in \{z|l(z) = 1\}$—assumed to be simple with $l$ holomorphic there—defines, by geometric transformation of its Taylor series, a *Laurent series* (Behnke and Sommer, 1962) and a residue

$$l(z) = 1 + (z-z_0)l'(z_0) + \sum_{k=2}^{\infty} \frac{(z-z_0)^k}{k!} f^{(k)}(z_0)$$

$$\frac{l(z)}{1-l(z)} = \frac{a-1(z_0)}{z-z_0} + \sum_{k=0}^{\infty}(z-z_0)^k a_k(z_0)$$

$$a_{-1}(z_0) = -\frac{1}{l'(z_0)}$$

Each eigenvalue $\{z_k|l(zk) = 1\}$ has its own principal part with the *Mittag–Leffer sum* replacing the simple pole for time or position

$$z \mapsto l(z) \mapsto \frac{l(z)}{1-l(z)} \mapsto \sum_{z_k} \frac{\alpha - 1(z_k)}{z - z_k}$$

The generalization for higher order poles is obvious.

Therewith one obtains the transition from the eigenvalue function $\tilde{l}_0 * \tilde{g}$ to complex representation functions for the Poincaré group

$$\frac{\tilde{l}_{0*}}{1-\tilde{l}_{0*}} : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}', \quad \begin{cases} \tilde{g} \mapsto \tilde{l}_0 * \tilde{g} \\ \tilde{l}_0 * \tilde{g}(q^2) = 1 + (q^2 - m^2)\frac{d}{dq^2}\tilde{l}_0 * \tilde{g}(m^2) + \cdots \\ \tilde{g}(q^2) \mapsto \frac{\tilde{l}_0 * \tilde{g}(q^2)}{1-\tilde{l}_0 * \tilde{g}(q^2)} = \frac{a_{-1}(m^2)}{q^2-m^2} + \cdots \\ -\frac{1}{a_{-1}(m^2)} = \frac{d}{dq^2}\tilde{l}_0 * \tilde{g}(m^2) \end{cases}$$

## 12.2. Residues as Coupling Constants

For the residual spacetime product above $(\frac{q}{q^2} \overset{\mathrm{R}}{*} \tilde{g})(q^2)$ the residual normalization $a_{-1}(0)$ for the massless solution $\tilde{l}_{\mathbf{1}}(0) = 1$ is given by the inverse of the negative derivative of the eigenvalue function there

$$-\frac{1}{a_{-1}(0)} = \frac{\partial}{\partial q^2}\tilde{l}_{\mathbf{1}}(0) = \begin{cases} \frac{1-m_3^2}{4m_3^2} = \frac{e^2-1}{4} \sim 1.6 & \text{for } \mathbb{R}^2 \\ \frac{1-6m_3^2+m_3^4}{12m_3^2} \sim \frac{e^5}{12} \sim 12.4 & \text{for } \mathbb{R}^4 \end{cases}$$

With the geometric transformation the principal part in the Laurent series gives an energy–momentum spacetime translation representation function for mass zero with residual normalization

$$\frac{\frac{q}{q^2}*}{1-\frac{q}{q^2}*} : \mathbb{R}^2_\vee \to \mathbf{SO}_0(1,1) \vec{\times} \mathbb{R}^2 : \int_{m_3^2}^1 dm^2 \frac{q}{(q^2-m^2)^2} \mapsto \mathbf{1}_2 \frac{a_{-1}(0)}{q^2} + \cdots$$

$$\frac{\frac{q}{q^2}*}{1-\frac{q}{q^2}*} : \mathbb{R}^4_\vee \to \mathbf{SO}_0(1,3) \vec{\times} \mathbb{R}^4 : \int_{m_3^2}^1 dm^2 \frac{(1-m^2)2q}{(q^2-m^2)^3} \mapsto \mathbf{1}_4 \frac{a_{-1}(0)}{q^2} + \cdots$$

With appropriate integration contour, it can be used as propagator for a mass zero spacetime vector field with coupling constant $-a_{-1}(0)$ which—with the signature $s-(d-s)$ only for 4-dimensional spacetime—has two particle interpretable degrees of freedom with a positive scalar product, related to the 2-sphere $\mathbb{R}^4_\vee/\mathbb{R}^2_\vee \cong \Omega^2$ with left and right axial $\mathbf{SO}(2)$-rotations (polarization)

$$-\eta^{jk} = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } \mathbf{SO}_0(1,1) \vec{\times} \mathbb{R}^2 \\ \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1}_2 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \text{for } \mathbf{SO}_0(1,3) \vec{\times} \mathbb{R}^4 \end{cases}$$

All the numerical results depend on the normalizations—trace normalization, dual normalization—which require a deeper understanding. If those normalizations can be trusted and if appropriate representations of the compact internal degrees of freedom for **U**(2) hypercharge and isospin are included, the residue of the arising propagator with mass zero in 4-dimensional spacetime may be compared with the coupling constant (Heisenberg, 1967) in the propagator of a massless gauge field, e.g., for the electromagnetic interaction and the left and right polarized photons with Sommerfeld's fine structure constant $\alpha$

$$\mathbf{SO}_0(1,3)\ \vec{\times}\ \mathbb{R}^4 : -\eta^{jk}\frac{e^2(0)}{q^2} \quad \text{with} \quad \frac{1}{e^2(0)} = \frac{1}{4\pi\alpha} \sim \frac{137}{12.6} \sim 10.9$$

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